Modelling Intentionality: The Gambler

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Introduction

• We have considered many models with autonomous individuals or *agents* (eg. birth/death, citizens of cities, ants).

• Agents have been modelled by assuming simple probability distribution for possible actions and treating as unmotivated, "dumb" particles (eg. birth/death rate, diffusion).

• Advantage: allowed analytical solutions via Master or Fokker-Planck equations.

• Disadvantage: often not very realistic.

• Introducing *intentional* motivated agents can complicate analysis but can be very rewarding [1, 2, 3].

• Simplest to make agents *perfectly rational* with complete knowledge and unlimited computational ability.

• Tends to produce simple, static equilibria (eg. ants make a bee-line straight for food).

• Aside: selfish motives can produce sub-optimal behaviour (eg. Braes' paradox [4], Diner's Dilemma [5])

Bounded Rationality

- Limited knowledge and/or computational ability

- Selfish motives

The Gambler's Ruin Paradox

- Related to St. Petersburg paradox [6]

- Gambler playing a "double or nothing" type game repeatedly against an infinitely rich adversary (multiplicative stochastic process).

- Chance of winning each time is p (known by gambler).

- In each iteration gambles a fraction r of wealth.

- Wealth after t iterations is W_t.

- Aside: can also be interpreted as a simple market model with one risky asset (price fluctuations as described above) and one riskless asset paying no interest. r describes portfolio.

- Expected wealth is

$$\begin{aligned}\langle W_t(r)\rangle &= p\,2r\,\langle W_{t-1}\rangle + (1-r)\,\langle W_{t-1}\rangle \\ &= ((2p-1)r+1)\,\langle W_{t-1}\rangle \\ &= ((2p-1)r+1)^t W_0\end{aligned}$$

• Naive goal is to maximize $\langle W_t \rangle$ w.r.t. r:

$$r^* = \begin{cases} 1 & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2} \\ \text{irrelevant} & \text{if } p = \frac{1}{2} \end{cases}$$

• If $p > \frac{1}{2}$ then "rational" gambler will wager everything.

• But must eventually lose (if $p < 1$):

$$\lim_{t\to\infty} P(W_t(1) > 0) = \lim_{t\to\infty} p^t = 0$$

• Problem arises from heavy weighting of *extremely* unlikely events.

• Expectation maximization is a poor choice for modelling rational behaviour. So what is "rational"?

Alternatives

Minimizing risk

• Maximizing expectation is too risky. Instead, might want to minimize risk of eventual ruin.

• If $r < 1$ then will never gambler will never lose *all* wealth so let's define *ruin* as a net loss $W_t < W_0$. (Could also define ruin by granularity of money (eg. *1 penny*) with the same conclusions.)

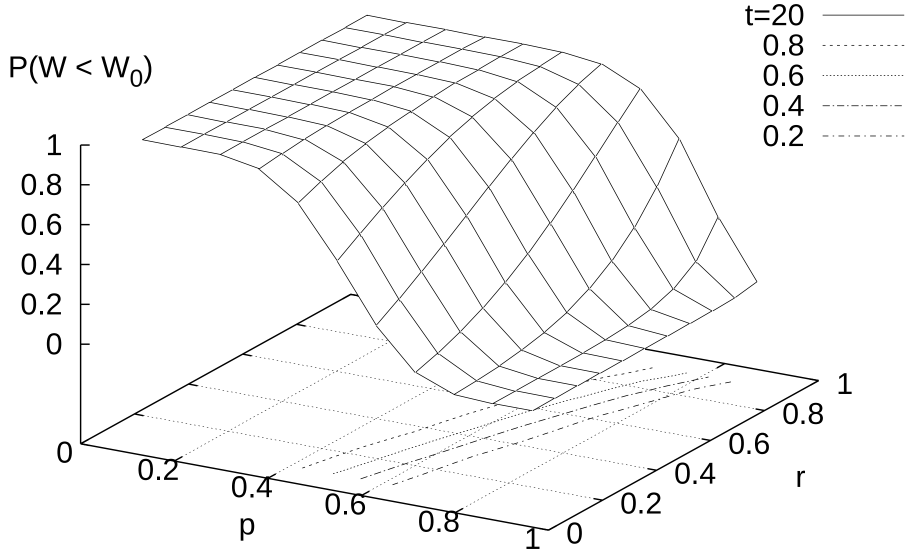
• Goal is to minimize $P(W_t(r) < W_0)$ (for t large).

• For large t wealth distribution is roughly log-normal (because random walk on log-scale; will be discussed later).

$$
\begin{aligned}
P(W < W_0) &= \int_0^{W_0} P(W)\, dW \\
&= \int_{-\infty}^{0} P(h)\, dh \\
&= \frac{1}{2}\mathrm{erfc}\left(v\sqrt{\frac{t}{2D}}\right)
\end{aligned}
$$

where $h = \ln(W/W_0)$ and $P(h)$ is normally distributed with mean vt and variance Dt (also to be discussed later).

• Solution becomes clear from the probability distribution itself (with v and D expanded in terms of p and r).



• Minimize risk by choosing

$$r^* = \begin{cases} 0 & \text{if } 0 < p < 1 \\ \text{irrelevant} & \text{if } p = 1 \\ 0 & \text{if } p = 0 \text{ (not shown, artifact)} \end{cases}$$

• So, in this interpretation, "rational" behaviour is to never gamble unless $p = 1$. Too safe?

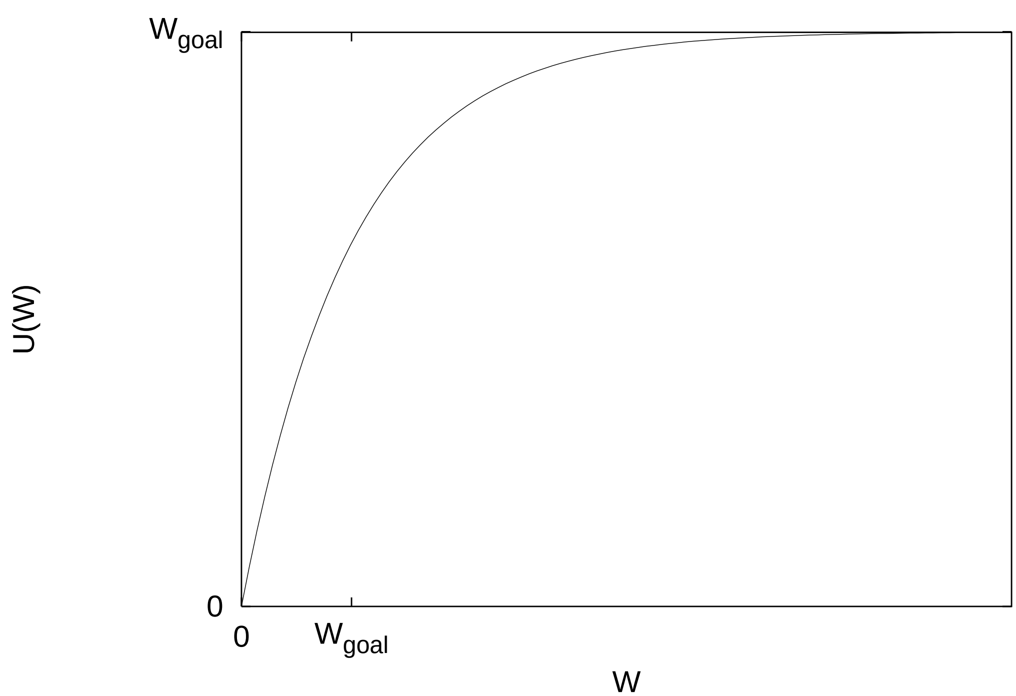
Utility function

• Agent values utility rather than wealth $U(W)$.

• Utility is increasing, concave function ($U' > 0$, $U'' < 0$). Literature suggests particulars of utility function largely irrelevant.

• A popular choice in finance is exponential utility $U_e(W) = -e^{-aW}$, or equivalently

$$U_e(W) = W_{goal}\left(1 - e^{-W/W_{goal}}\right)$$



• W_{goal} can be interpreted as maximum *conceivable* wealth or goal wealth (determines riskyness). (For finite system W_{goal}

must be not be greater than all available wealth.)

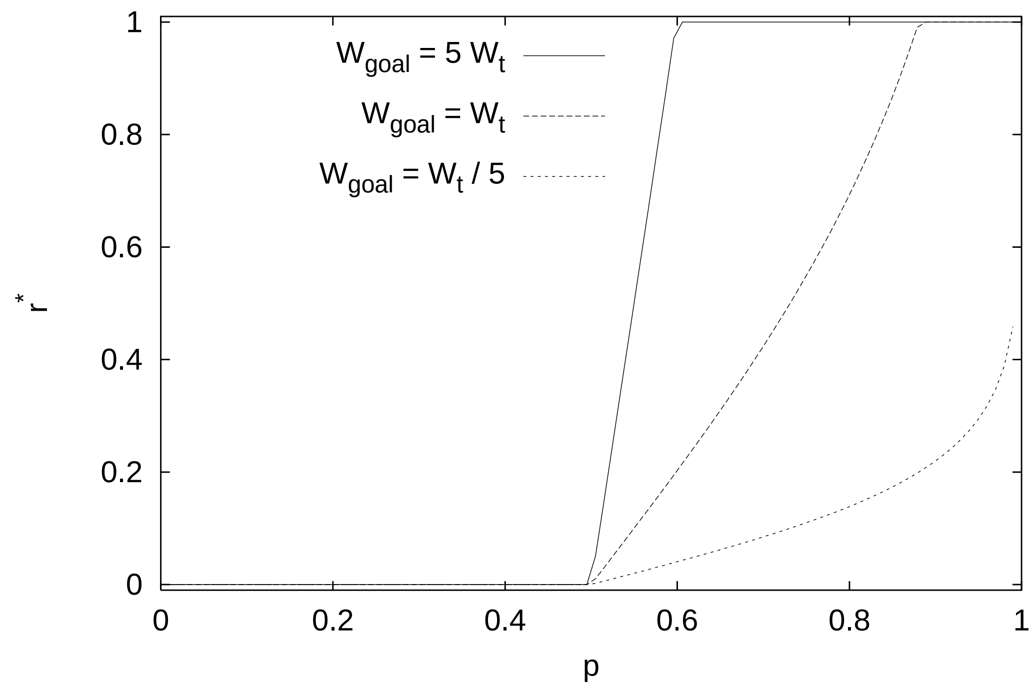
• Consider a single iteration. Goal is to maximize expected utility

$$\langle U(W_{t+1}(r))\rangle = p\,U((1+r)W_t) + (1-p)\,U((1-r)W_t)$$

• Solution: optimal investment fraction r^* is

$$r^* = \frac{W_{goal}}{2W_t}\ln\left(\frac{p}{1-p}\right)$$

• r^* changes with each iteration as W_t changes. Decreases as wealth increases to W_{goal}.



• Non-trivial solution for $1/2 < p < 1$.

• Gamblers are still “irrational” because they will always gamble their entire wealth ($r^* = 1$) when the chance of winning is greater than

$$p_{sucker} = \frac{1}{1 + e^{-2W_t/W_{goal}}} < 1$$

Kelly Utility

• Also common in the literature is the *generalized Kelly utility* [7]

$$U_k(W) = \begin{cases} \frac{W^{1-1/k}}{1-1/k} & \text{if } k \neq 1 \\ \ln W & \text{if } k = 1 \end{cases}$$

• $k = 1$ utility equivalent to $k \to 1$ because derivatives $\partial_W U$ the same. Absolute value doesn't matter for optimization.

• Kelly [8] originally hypothesized just the logarithmic form. Was generalized to $k \neq 1$ later.

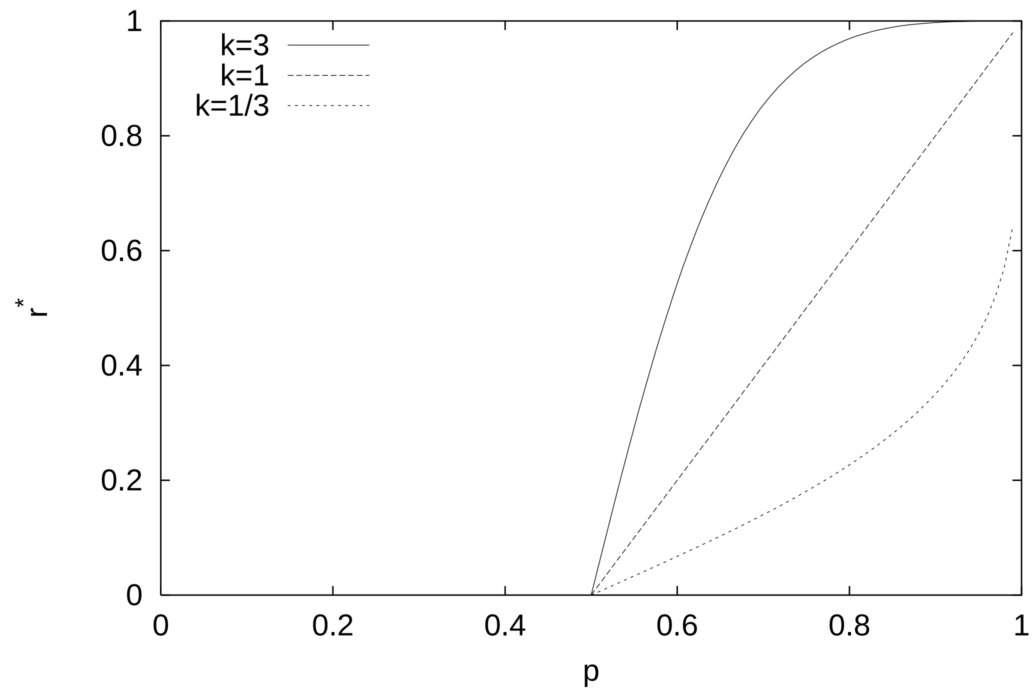
• The advantage over previous utility is that there is no arbitrary cut-off wealth W_{goal}, but there is a parameter k (the Kelly parameter). Meaning will become clear.

• Again, goal is to maximize expected utility

$$\langle U(W_{t+1}(r))\rangle = p\,U((1+r)W_t) + (1-p)\,U((1-r)W_t)$$

which gives

$$r^* = \frac{p^k - (1-p)^k}{p^k + (1-p)^k}$$



• Kelly parameter k is "riskyness". $k < 1$ = risk-adverse, $k > 1$ = risk-prone. $1/k$ is "risk aversion".

• Kelly utility is more "rational" because $r^* = 1$ iff $p = 1$.

Median value [9, 10]

• Perhaps using the expectation value is an unfortunate choice. Often the median value is a more typical realization. Then a rational goal might be to try and optimize the median value of the future wealth.

• Median W_{med} is defined as point with equal probability of greater or lesser values:

$$P(W > W_{med}) = P(W < W_{med}) = 1/2$$

• To derive the median value we must recognize that the wealth W_t follows a multiplicative random walk

$$\begin{aligned} W_{t+1}(r) &= (1 \pm r)W_t(r) \\ &= e^{\eta_t} W_t(r) \end{aligned}$$

where η is distributed via

$$\pi(\eta) = p\,\delta(\eta - \ln(1+r)) + (1-p)\,\delta(\eta - \ln(1-r))$$

• Use log-scale to get additive noise

$$h_t = \ln W_t$$

$$h_{t+1} = h_t + \eta_t$$

• Random (biased) walk so, after many iterations, $P(h, t)$ approaches a Gaussian distribution with drift velocity v and dispersion D

$$v = \langle \eta \rangle = p \ln(1+r) + (1-p) \ln(1-r)$$

$$\begin{aligned} D &= \left\langle \eta^2 \right\rangle - \langle \eta \rangle^2 \\ &= p(1-p) \ln^2 \left(\frac{1+r}{1-r} \right) \end{aligned}$$

• Median (and average) of h linear in time $h_{med} = vt$.

• Median of wealth is

$$W_{med} = W_0 e^{h_{med}}$$

by definition, because

$$\frac{1}{2} = \int_{h_{med}}^{\infty} P(h) dh = \int_{W_{med}=W(h_{med})}^{\infty} P(W) dW$$

• Goal is to maximize median (typical) wealth W_{med} w.r.t. r

$$\begin{aligned} 0 &= \partial_r W_{med} \\ &= W_{med} \, \partial_r h_{med} \\ &= W_{med} \, t \, \partial_r v \end{aligned}$$

which has solution

$$r^* = 2p - 1$$

• Same solution we saw for the Kelly utility function ($k = 1$). Optimizing Kelly utility equivalent to optimizing median value.

Comparison

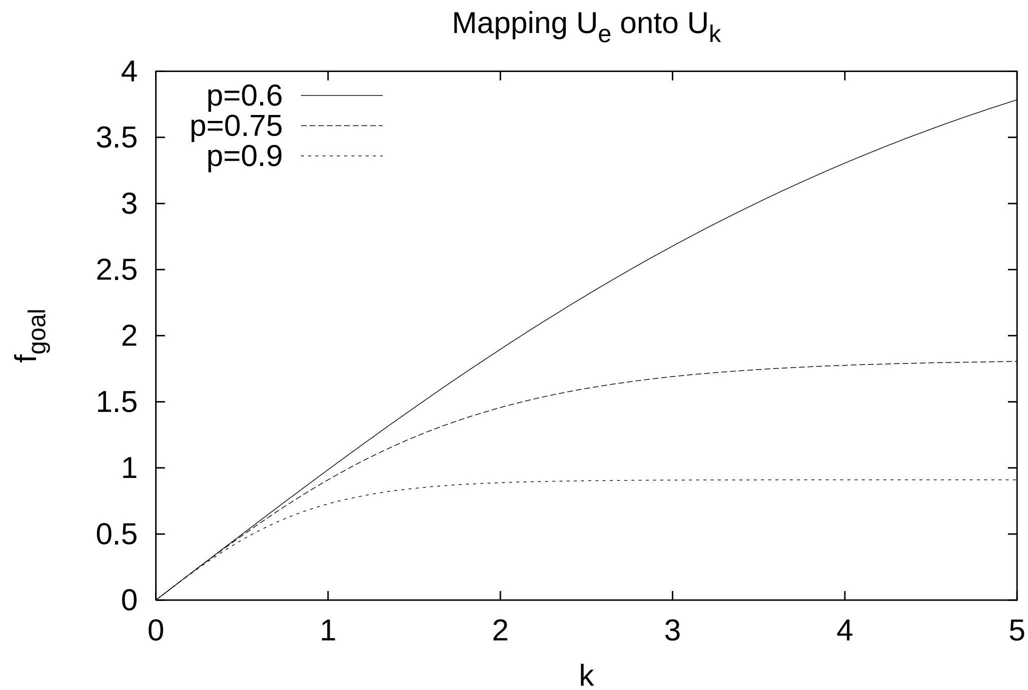
Mapping

• For exponential utility, must update r^* with each iteration. If we instead update W_{goal} via $W_{goal} = f_{goal} W_t$ for some multiplier f_{goal} then r^* constant.

$$r^* = \frac{f_{goal}}{2} \ln \left(\frac{p}{1-p} \right)$$

• Can map exponential utility U_e onto Kelly utility U_k by equating r^* yielding

$$f_{goal} = 2 \frac{p^k - (1-p)^k}{(p^k + (1-p)^k) \ln \frac{p}{1-p}}$$



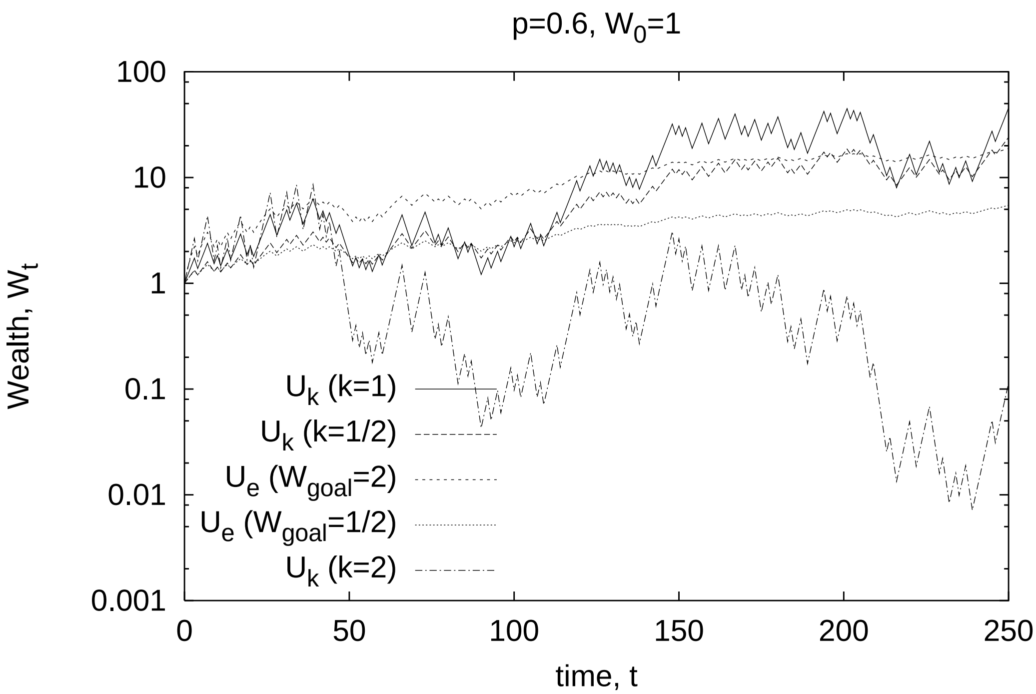
• Aside: regardless of riskyness k, for $p > 1/2$, f_{goal} is bounded in order that agent remain "rational" $(r^* < 1)$

$$f_{max} = \lim_{k \to \infty} f_{goal} = \frac{2}{\ln \frac{p}{1-p}}$$

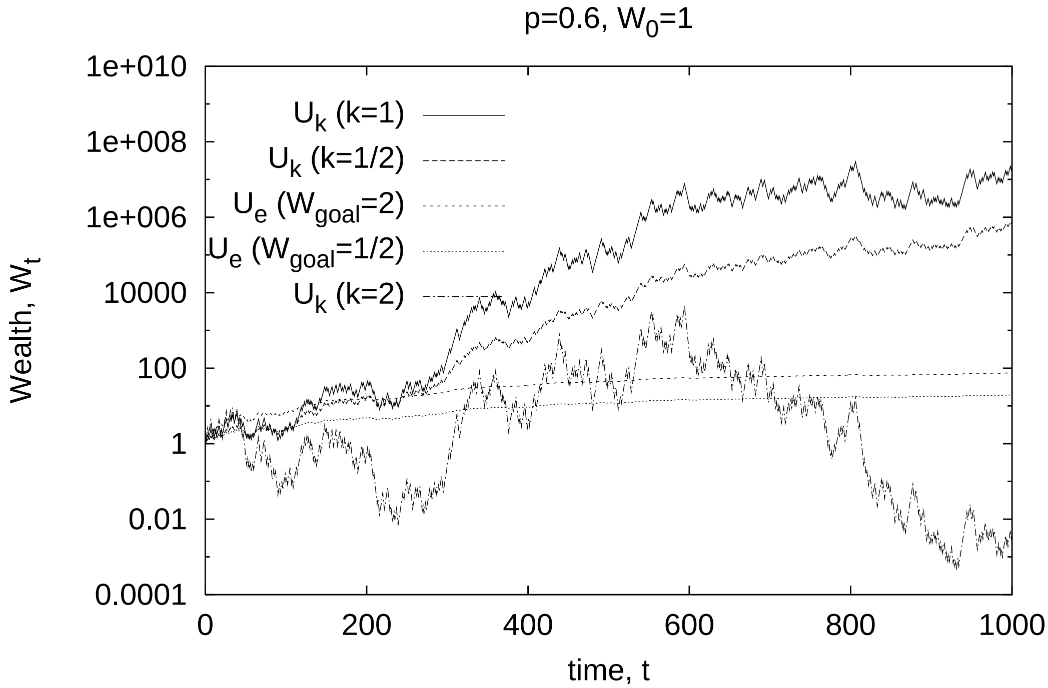
Simulation

• All approaches explored (neglecting trivial solutions) so far reduce to two models: U_e (W_{goal}=constant) and U_k.

• Simulation shows iterated wealth of two exponential utilities (W_{goal} = 1/2 and 2) and three Kelly utilities (k = 1/2, 1 (log) and 2). Used p=0.6 and W_0=1 and all agents used same history of wins/losses.



• All but risky k=2 Kelly utility performed well over short term



• Fixed W_{goal} exponential utilities underperform (as expected) over long time. They can also crash (W_t=0) if wealth gets too low (not seen in this realization).

• k=1 (log) has best performance long-term but safe k=1/2 also good.

Conclusions

• Asked question "How do we model *rational* agents?"

• Looked at gambler playing a "double-or-nothing" type game.

• Tricky because multiplicative process.

• Maximizing expectation too risky.

• Minimizing risk too safe.

• Common (exponential) utility can be too risky (and contains arbitrary scale).

• Generalized Kelly utility favourable.

• Maximizing median value equivalent to (original) Kelly utility.

• Median and expected values can be very different in multiplicative processes.

• Simulations suggest optimizing median value best. But smaller Kelly number can be just as good (and safer) on short timescales.

• Maslov and Zhang [11] proved $k{=}1$ is on borderline of riskiness for similar model.

• Exercise: Prove Kelly utility ($k = 1$) optimal as $t \to \infty$ [12].

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