Assignment #5

Chapter 8: 16, 19
Chapter 9: 19

Due this Friday Oct. 30th by 2pm in your TA’s homework box
Assignment #6

Chapter 10: 14, 15
Chapter 11: 14, 18

Due Next Friday Nov. 6\textsuperscript{th} by 2pm in your TA’s homework box
Reading

For Today: Chapter 11 & 12

For Thursday: Chapter 12
First Part of Chapter 11 Review
$\mu = 67.4$

$\sigma = 3.9$

$\bar{Y}$ is normally distributed whenever:

1. $Y$ is normally distributed
2. $n$ is large

$\mu_{\bar{Y}} = \mu = 67.4$

$\sigma_{\bar{Y}} = \frac{\sigma}{\sqrt{n}} = \frac{3.9}{\sqrt{5}} = 1.7$
Inference about means

Because $\overline{Y}$ is normally distributed, we can convert its distribution to a standard normal distribution:

$$Z = \frac{\overline{Y} - \mu}{\sigma_{\overline{Y}}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

This would give a probability distribution of the difference between a sample mean and the population mean.
But... We don’t know $\sigma$...

However, we do know $s$, the standard deviation of our sample. We can use that as an estimate of $\sigma$. 
In most cases, we don’t know the real population distribution.

We only have a sample.

\[ \mu = 67.4 \]
\[ \sigma = 3.9 \]

\[ \bar{Y} = 67.1 \]
\[ s = 3.1 \]

\[ SE_{\bar{Y}} = \frac{s}{\sqrt{n}} = \frac{3.1}{\sqrt{5}} = 1.4 \]

We use this as an estimate of \( \sigma_{\bar{Y}} \)
A good approximation to the standard normal is then:

\[ t = \frac{\bar{Y} - \mu}{SE_{\bar{Y}}} = \frac{\bar{Y} - \mu}{s/\sqrt{n}} \]
$t$ has a Student’s $t$ distribution

$Z = \frac{\bar{Y} - \mu}{\sigma_{\bar{Y}}}$

$t = \frac{\bar{Y} - \mu}{SE_{\bar{Y}}}$

Discovered by William Gossett, of the Guinness Brewing Company
Degrees of freedom

\[ df = n - 1 \]
We use the $t$-distribution to calculate an exact confidence interval of the mean

$$-t_{\alpha(2), df} < \frac{\bar{Y} - \mu}{SE_{\bar{Y}}} < t_{\alpha(2), df}$$

We rearrange the above to generate:

$$\bar{Y} - t_{\alpha(2), df} SE_{\bar{Y}} < \mu < \bar{Y} + t_{\alpha(2), df} SE_{\bar{Y}}$$

Another way to express this is: $$\bar{Y} \pm SE_{\bar{Y}} t_{\alpha(2), df}$$
Example

Are species ranges shifting towards higher elevations as the world warms? Highest elevation shift (m) over late 1900s and early 2000s was measured for 31 species.

\[ \bar{Y} = 39.329 \]
\[ s = 30.663 \]
\[ n = 31 \]
Find the standard error

\[
\bar{Y} \pm SE_{\bar{Y}} t_{\alpha(2), df}
\]

\[
SE_{\bar{Y}} = \frac{s}{\sqrt{n}} = \frac{30.663}{\sqrt{31}} = 5.507
\]
Find the critical value of $t$

$$df = n - 1 = 31 - 1 = 30$$

$$t_{\alpha(2),df} = t_{0.05(2),30} = 2.04$$
Table C: Student's $t$ distribution

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Putting it all together...

\[
\bar{Y} \pm SE_{\bar{Y}} t_{\alpha(2), df} = 39.329 \pm 5.507 \left(2.04\right)
\]
\[
= 39.329 \pm 11.234
\]

\[
28.09 < \mu < 50.56
\]

(95% confidence interval)
99% confidence interval

\[ t_{\alpha(2), df} = t_{0.01(2), 30} = 2.75 \]

\[ \bar{Y} \pm SE_{\bar{Y}} t_{\alpha(2), df} = 39.329 \pm 5.507 (2.75) \]

\[ = 39.329 \pm 15.144 \]

\[ 24.185 < \mu < 54.473 \]
One-sample $t$-test

The *one-sample* $t$-test compares the mean of a random sample from a normal population with the population mean proposed in a null hypothesis.
One-sample $t$-test:
Assumptions

- The variable is normally distributed.
- The sample is a random sample.
Test statistic for one-sample $t$-test

$$t = \frac{\bar{Y} - \mu_0}{s / \sqrt{n}}$$

$\mu_0$ is the mean value proposed by $H_0$
Hypotheses for one-sample \( t \)-tests

\[ H_0 : \text{The mean of the population is } \mu_0. \]

\[ H_A : \text{The mean of the population is not } \mu_0. \]
Example

Are species ranges shifting towards higher elevations as the world warms? Highest elevation shift (m) over late 1900 and early 2000s was measured for 31 species.

\[ \bar{Y} = 39.329 \]
\[ s = 30.663 \]
\[ n = 31 \]
Example

$H_0$: The mean elevation change is 0.

$H_A$: The mean elevation change is not 0.
Example

\[ n = 31 \]
\[ \bar{Y} = 39.329 \]
\[ s / \sqrt{n} = 5.507 \]

\[ t = \frac{\bar{Y} - \mu_0}{s / \sqrt{n}} = \frac{39.329 - 0}{5.507} = 7.141 \]
Table C: Student's $t$ distribution

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<td>1.71</td>
<td>2.06</td>
<td>2.49</td>
<td>2.80</td>
<td>3.75</td>
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<tr>
<td>25</td>
<td>1.32</td>
<td>1.71</td>
<td>2.06</td>
<td>2.49</td>
<td>2.79</td>
<td>3.73</td>
<td>4.62</td>
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<tr>
<td>26</td>
<td>1.31</td>
<td>1.71</td>
<td>2.06</td>
<td>2.48</td>
<td>2.78</td>
<td>3.71</td>
<td>4.59</td>
<td></td>
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<tr>
<td>27</td>
<td>1.31</td>
<td>1.70</td>
<td>2.05</td>
<td>2.47</td>
<td>2.77</td>
<td>3.69</td>
<td>4.56</td>
<td></td>
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<tr>
<td>28</td>
<td>1.31</td>
<td>1.70</td>
<td>2.05</td>
<td>2.47</td>
<td>2.76</td>
<td>3.67</td>
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<tr>
<td>29</td>
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<td>1.70</td>
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<td>4.51</td>
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<tr>
<td>30</td>
<td>1.31</td>
<td>1.70</td>
<td>2.04</td>
<td>2.46</td>
<td>2.75</td>
<td>3.65</td>
<td>4.48</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example

\[ t = 7.141 \]

\[ t_{0.05(2),30} = \pm 2.04 \]

\[ t_{0.0001(2),30} = \pm 4.48 \]

\( t \) is further out in the tail than the critical value, so we reject the null hypothesis. Species have shifted to higher elevations. \( P < 0.0001 \)
Inference from a normal population

Chapter 11 Continued
Confidence interval for the variance

\[
\frac{df}{\chi^2_{\frac{\alpha}{2}, df}} s^2 \leq \sigma^2 \leq \frac{df}{\chi^2_{1-\frac{\alpha}{2}, df}} s^2
\]
95% confidence interval for a variance

Example:
Paradise flying snakes

Undulation rates (in Hz)

0.9, 1.4, 1.2, 1.2, 1.3, 2.0, 1.4, 1.6
\[ \bar{Y} = 1.375 \]
\[ s = 0.324 \]
\[ n = 8 \]
95% confidence interval for the variance of flying snake undulation rate

\[
\frac{df \cdot s^2}{\chi^2_{\frac{\alpha}{2}, df}} \leq \sigma^2 \leq \frac{df \cdot s^2}{\chi^2_{1-\frac{\alpha}{2}, df}}
\]

df = n - 1 = 7

s^2 = (0.324)^2 = 0.105
\[ x_{\alpha, \frac{2}{2}, df}^2 = x_{0.025, 7}^2 = 16.01 \]

\[ x_{1-\alpha, \frac{2}{2}, df}^2 = x_{0.975, 7}^2 = 1.69 \]

Table A

<table>
<thead>
<tr>
<th>df</th>
<th>( x )</th>
<th>0.999</th>
<th>0.995</th>
<th>0.99</th>
<th>0.975</th>
<th>0.95</th>
<th>0.05</th>
<th>0.025</th>
<th>0.01</th>
<th>0.005</th>
<th>0.001</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1.6 E-6</td>
<td>3.9E-5</td>
<td>0.00016</td>
<td>0.00098</td>
<td>0.00393</td>
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<td>5.02</td>
<td>6.63</td>
<td>7.88</td>
<td>10.83</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.01</td>
<td>0.02</td>
<td>0.05</td>
<td>0.1</td>
<td>5.99</td>
<td>7.38</td>
<td>9.21</td>
<td>10.6</td>
<td>13.82</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.02</td>
<td>0.07</td>
<td>0.11</td>
<td>0.22</td>
<td>0.35</td>
<td>7.81</td>
<td>9.35</td>
<td>11.34</td>
<td>12.84</td>
<td>16.27</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.09</td>
<td>0.21</td>
<td>0.3</td>
<td>0.48</td>
<td>0.71</td>
<td>9.49</td>
<td>11.14</td>
<td>13.28</td>
<td>14.86</td>
<td>18.47</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.21</td>
<td>0.41</td>
<td>0.55</td>
<td>0.83</td>
<td>1.15</td>
<td>11.07</td>
<td>12.83</td>
<td>15.09</td>
<td>16.75</td>
<td>20.52</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.38</td>
<td>0.68</td>
<td>0.87</td>
<td>1.24</td>
<td>1.64</td>
<td>12.59</td>
<td>14.45</td>
<td>16.81</td>
<td>18.55</td>
<td>22.46</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.6</td>
<td>0.99</td>
<td>1.24</td>
<td>1.69</td>
<td>2.17</td>
<td>14.07</td>
<td>16.01</td>
<td>18.48</td>
<td>20.28</td>
<td>24.32</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.86</td>
<td>1.34</td>
<td>1.65</td>
<td>2.18</td>
<td>2.73</td>
<td>15.51</td>
<td>17.53</td>
<td>20.09</td>
<td>21.95</td>
<td>26.12</td>
<td></td>
</tr>
</tbody>
</table>
95% confidence interval for the variance of flying snake undulation rate

$$\frac{df}{\chi^2_{\alpha/2, df}} \leq \sigma^2 \leq \frac{df}{\chi^2_{1-\alpha/2, df}}$$

$$\frac{7 \times (0.324)^2}{16.01} \leq \sigma^2 \leq \frac{7 \times (0.324)^2}{1.69}$$

$$0.0459 \leq \sigma^2 \leq 0.435$$
95% confidence interval for the standard deviation of flying snake undulation rate

\[
\sqrt{\frac{df}{\chi_{\alpha/2, df}^2}} \leq \sigma \leq \sqrt{\frac{df}{\chi_{1-\alpha/2, df}^2}}
\]

\[
\sqrt{0.0459} \leq \sigma \leq \sqrt{0.435}
\]

\[
0.21 \leq \sigma \leq 0.66
\]
Comparing means

• Tests with one categorical and one numerical variable

• Goal: to compare the mean of a numerical variable for different groups.
Paired vs. 2 sample comparisons
Paired comparisons allow us to account for a lot of extraneous variation

2-sample methods are sometimes easier to collect data for
Paired designs

- Data from the two groups are paired
- Each member of the pair shares much in common with the other, except for the tested categorical variable
- There is a one-to-one correspondence between the individuals in the two groups
Paired design: Examples

• Before and after treatment

• Upstream and downstream of a power plant

• Identical twins: one with a treatment and one without
Paired comparisons

• We have many pairs

• In each pair, there is one member that has one treatment and another who has another treatment

(“Treatment” can mean “group”)
Paired comparisons

• To compare two groups, we use the mean of the *difference* between the two members of each pair.
Estimating difference in means from paired data

\[ \bar{d} = \text{Mean of differences between pairs} \]

\[ SE_{\bar{d}} = \frac{s_d}{\sqrt{n}} \]

\[ \bar{d} - t_{\alpha(2), df} SE_{\bar{d}} < \mu_d < \bar{d} + t_{\alpha(2), df} SE_{\bar{d}} \]
Example: National No Smoking Day

- Data compares injuries at work on National No Smoking Day (in Britain) to the same day the week before

- Each data point is a year

<table>
<thead>
<tr>
<th>Year</th>
<th>Injuries before No Smoking Day</th>
<th>Injuries on No Smoking Day</th>
</tr>
</thead>
<tbody>
<tr>
<td>1987</td>
<td>516</td>
<td>540</td>
</tr>
<tr>
<td>1988</td>
<td>610</td>
<td>620</td>
</tr>
<tr>
<td>1989</td>
<td>581</td>
<td>599</td>
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<tr>
<td>1990</td>
<td>586</td>
<td>639</td>
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<tr>
<td>1991</td>
<td>554</td>
<td>607</td>
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<tr>
<td>1992</td>
<td>632</td>
<td>603</td>
</tr>
<tr>
<td>1993</td>
<td>479</td>
<td>519</td>
</tr>
<tr>
<td>1994</td>
<td>583</td>
<td>560</td>
</tr>
<tr>
<td>1995</td>
<td>445</td>
<td>515</td>
</tr>
<tr>
<td>1996</td>
<td>522</td>
<td>556</td>
</tr>
</tbody>
</table>
Calculate differences

<table>
<thead>
<tr>
<th>Injuries before No Smoking Day</th>
<th>Injuries on No Smoking Day</th>
<th>Difference (d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>516</td>
<td>540</td>
<td>24</td>
</tr>
<tr>
<td>610</td>
<td>620</td>
<td>10</td>
</tr>
<tr>
<td>581</td>
<td>599</td>
<td>18</td>
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<tr>
<td>586</td>
<td>639</td>
<td>53</td>
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<tr>
<td>554</td>
<td>607</td>
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<td>479</td>
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<td>560</td>
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<tr>
<td>445</td>
<td>515</td>
<td>70</td>
</tr>
<tr>
<td>522</td>
<td>556</td>
<td>34</td>
</tr>
</tbody>
</table>
Estimating difference in means from paired data

\[
\begin{align*}
\bar{d} &= 25 \\
S_d &= 32.3 \\
n &= 10 \\
SE_{\bar{d}} &= \frac{32.3}{\sqrt{10}} = 10.2 \\
t_{\alpha(2),9} &= 2.26 \\
25 - 2.26(10.2) < \mu_d < 25 + 2.26(10.2) \\
1.948 < \mu_d < 48.052
\end{align*}
\]
Paired $t$ test

- Compares the mean of the differences to a value given in the null hypothesis

- For each pair, calculate the difference. The paired $t$-test is simply a one-sample $t$-test on the differences.
Hypotheses

$H_0$: Work related injuries do not change during No Smoking Days. ($\mu_d = 0$)

$H_A$: Work related injuries change during No Smoking Days. ($\mu_d \neq 0$)
Calculate $t$ using $d'$ s

\[
\bar{d} = 25
\]
\[
s_d^2 = 1043.78
\]
\[
n = 10
\]
\[
t = \frac{25 - 0}{\sqrt{1043.78/10}} = 2.45
\]
CAUTION!

- The number of data points in a paired $t$ test is the number of *pairs*. -- *Not* the number of individuals

- Degrees of freedom = Number of pairs - 1
So we can reject the null hypothesis. Stopping smoking increases job-related accidents in the short term.
Assumptions of paired \( t \) test

- Pairs are chosen at random
- The differences have a normal distribution

It does \textit{not} assume that the individual values are normally distributed, only the differences.
Estimating difference in means from two sample data

\[
\bar{Y}_1 - \bar{Y}_2
\]

Confidence interval: \( \left( \bar{Y}_1 - \bar{Y}_2 \right) \pm SE_{\bar{Y}_1 - \bar{Y}_2} t_{\alpha(2), df} \)
Standard error of difference in means

\[ SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} \]

Pooled variance:

\[ s_p^2 = \frac{df_1 s_1^2 + df_2 s_2^2}{df_1 + df_2} \]

\[ df_1 = n_1 - 1; \quad df_2 = n_2 - 1 \]
Costs of resistance to aphids

2 genotypes of lettuce: *Susceptible* and *Resistant*

Do these genotypes differ in fitness in the absence of aphids?
Both distributions are approximately normal.

<table>
<thead>
<tr>
<th></th>
<th>Susceptible</th>
<th>Resistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean number of buds</td>
<td>720</td>
<td>582</td>
</tr>
<tr>
<td>SD of number of buds</td>
<td>223.6</td>
<td>277.3</td>
</tr>
<tr>
<td>Sample size</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>
Calculating the standard error

\[ df_1 = 15 - 1 = 14; \quad df_2 = 16 - 1 = 15 \]

\[ s_p^2 = \frac{df_1 s_1^2 + df_2 s_2^2}{df_1 + df_2} = \frac{14(223.6)^2 + 15(277.3)^2}{14 + 15} = 63909.9 \]

\[ SE_{\bar{x}_1 - \bar{x}_2} = \sqrt{s_p^2} = \sqrt{\frac{63909.9}{15} + \frac{63909.9}{16}} = 90.86 \]
Finding $t$

$$df = df_1 + df_2 = n_1 + n_2 - 2$$

$$= 15 + 16 - 2$$

$$= 29$$

$$t_{0.05(2), 29} = 2.05$$
The 95% confidence interval of the difference in the means

\[
\left( \bar{Y}_1 - \bar{Y}_2 \right) \pm SE_{\bar{Y}_1 - \bar{Y}_2} t_{\alpha(2),df} = (720 - 582) \pm 90.86(2.05)
\]

\[= 138 \pm 186\]
Testing hypotheses about the difference in two means

2-sample $t$-test

The *two sample* $t$-test compares the means of a numerical variable between two populations.
Hypotheses

$H_0$: There is no difference between the number of buds in the susceptible and resistant plants. ($\mu_1 = \mu_2$)

$H_A$: The resistant and the susceptible plants differ in their mean number of buds. ($\mu_1 \neq \mu_2$)
2-sample t-test

t = \frac{\bar{Y}_1 - \bar{Y}_2}{SE \frac{\bar{y}_1 - \bar{y}_2}{\bar{y}_1 - \bar{y}_2}}
2-sample t-test
More Generally:

\[
t = \frac{(\overline{Y}_1 - \overline{Y}_2) - (\mu_1 - \mu_2)_0}{SE_{\overline{Y}_1 - \overline{Y}_2}}
\]

\[(\mu_1 - \mu_2)_0 = 0\]
Calculating $t$

\[ t = \frac{(\bar{Y}_1 - \bar{Y}_2)}{SE_{\bar{Y}_1 - \bar{Y}_2}} = \frac{(720 - 582)}{90.86} = 1.52 \]
Drawing conclusions...

$t_{0.05(2),29} = 2.05$

t < 2.05, so we cannot reject the null hypothesis.

These data are not sufficient to say that there is a cost of resistance.
Assumptions of two-sample $t$-tests

- Both samples are random samples.
- Both populations have normal distributions
- The variance of both populations is equal.