Biomathematics 301 Midterm 2019

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Student #____________________________________

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Question 1 [40 points]: In a recent paper, Erten et al. (2018) studied shifts in culture, focusing on the immigration of a cultural trait of interest from a location where that trait is common into a new area.

> **FROM THEIR ABSTRACT (READ ONLY IF YOU'RE CURIOUS AND HAVE TIME):**
> "The migration of people between different cultures has affected cultural change throughout history… Our study demonstrates that formal dynamic models can help us understand how individual orientations towards immigration eventually determine the population-level distribution of cultural traits." *Nature Communications* 9:58

Their equation (1) can be simplified to:

$$p[t+1] = (1-m)\,p[t] + m + c\,p[t]\,(1-p[t])$$

where $p[t]$ is the frequency of the immigrant cultural trait in the study population ($0 \le p[t] \le 1$), m is the proportion of immigrants, assumed to carry the trait ($0 < m < 1$), and c is the rate at which individuals holding different cultural traits interact times the probability that an interaction leads to locals adopting the immigrant trait ($c > 0$) or immigrants adopting the local trait ($c < 0$).

(a) This equation describes (tick all that apply):

- X Discrete-time model
- X Recursion equation
- ☐ Equilibrium equation
- ☐ Continuous-time model
- ☐ Differential equation
- ☐ General solution

(b) Determine the equilibria of the model:

$\hat{p} = (1-m)\hat{p} + m + c\,\hat{p}(1-\hat{p})$ Subtracting $\hat{p}$ from both sides and combine the two terms with m:

$0 = m(1-\hat{p}) + c\,\hat{p}(1-\hat{p})$ Factor out a (1-$\hat{p}$):

$0 = (m + c\,\hat{p})(1-\hat{p})$ Solve to get the two equilibria: $\hat{p} = 1$ and $\hat{p} = -m/c$

(c) For each equilibrium, say when it will be biologically valid given the definition of p and the constraints on the parameters.

Given that $\hat{p}$ is a frequency, $\hat{p} = 1$ is always valid. $\hat{p} = -m/c$ is positive when $c < 0$ (recall that $0 < m < 1$) and less than one when $-\frac{m}{c} < 1$. Put together, because $c < 0$, this requires that $m < -c$ or $c < -m$ [Careful not to multiply both sides of the inequality by a negative number, c, without flipping the inequality.]

(d) Determine when the equilibrium at $\hat{p} = 1$ (immigrant trait fixed) will be locally stable.

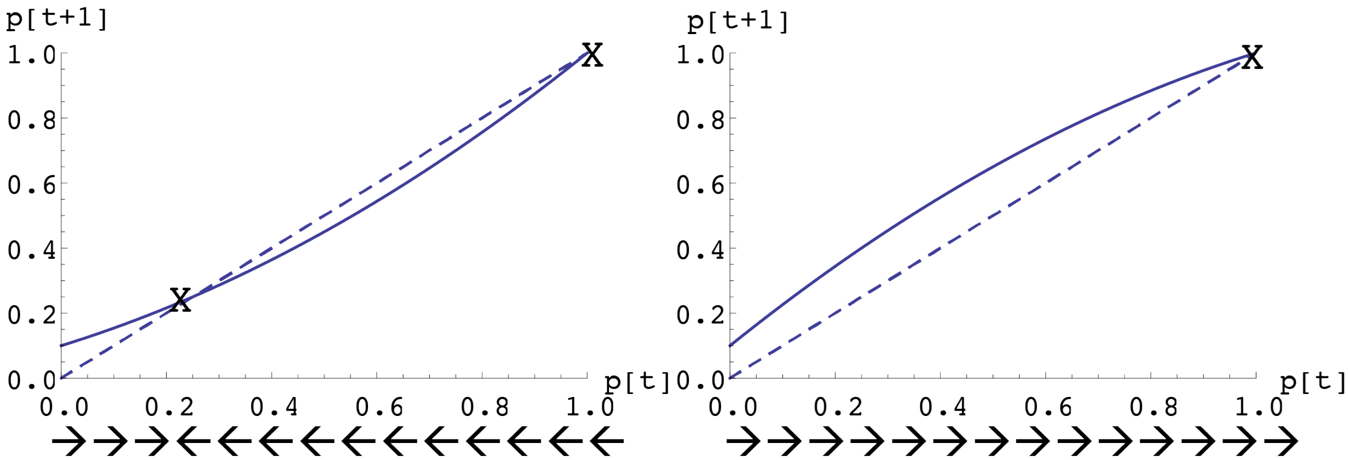
$\frac{df}{dp} = (1 - m) + c - 2\,c\,p[t]$ so $\left.\frac{df}{dp}\right|_{\hat{p}=1} = (1 - m) - c$.

To be stable, $-1 < 1 - m - c < 1$

or $-2 < -m - c < 0$

or $-2 + m < -c < m$

(e) In the graphs below of the p[t+1] versus p[t] (solid curves), add an X to indicate each equilibrium and add arrows along the horizontal axis to indicate when p[t] increases (→) and when p decreases (←).



(f) Based on these graphs, answer the following **[CIRCLE EITHER TRUE OR FALSE]**:

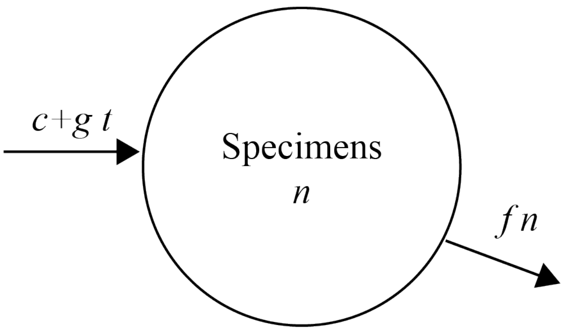
True / False: There is a single globally stable equilibrium in both graphs.

True / False: Both the immigrant and local cultural trait are always present at equilibrium.

True / False: The immigrant trait always spreads when rare.

Question 2 [50 points]: The collections in the Beaty Museum are constantly changing. Because the reputation of the Beaty is growing over time, *t*, the rate of new specimens being donated is increasing; we will model the rate of input of new specimens as $c + g\ t$, where *c* and *g* are positive constants. On the other hand, specimens do break and degrade at a rate, *f*, and must be thrown out. We could use the differential equation $\frac{dn}{dt} = -f\ n\ + c + g\ t$ to describe the number of specimens in the collection, *n*.

(a) Draw a flow diagram for the number of specimens, *n*, over time. Place flow rates over each arrow.



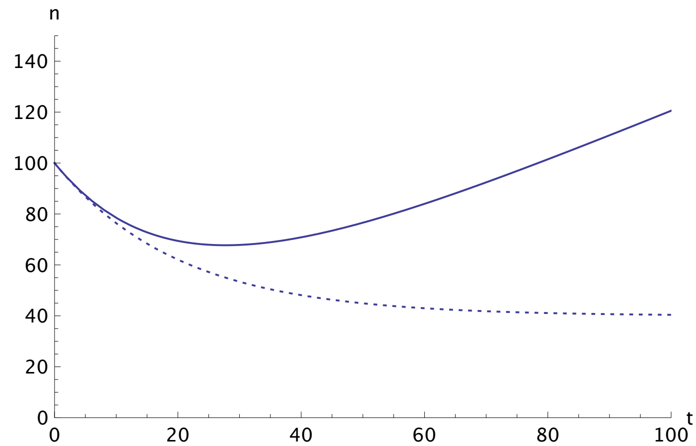
(b) State whether this equation can be solved using a separation of variables **[CIRCLE EITHER YES OR NO]**. If yes, begin the separation of variables by rewriting the differential equation with the variables separated. **[Do not carry out the integration!]**

YES / NO: $\int$ $= \int$

(c) The general solution to this equation is $n = e^{-f\ t}\left(\frac{g-f\ c}{f^2} + n_0\right) + \frac{f\ c+f\ g\ t - g}{f^2}$, where n_0 is the initial number of specimens at $t = 0$. According to this solution, describe what happens to the system over the long term, as *t* gets very large.

As the parameters are positive, $e^{-f\ t}$ will go to zero, and $\frac{f\ c+f\ g\ t - g}{f^2}$ will become dominated by the *t* term, rising as $\frac{f\ g\ t}{f^2} = \frac{g\ t}{f}$. Ok to say that it goes to infinity or that it rises linearly with time (assuming g>0).

(d) The following plot shows curves for the general solution over time, one assuming $g = 0$ and one assuming $g = 0.05$. Label each curve (write what g equals by the curve) and briefly explain your logic. [The remaining parameters were $n_0 = 100$, $f = 0.05$, and $c = 2$.]



Explain your choice:

$g = 0.05$ is the top curve and 0 the bottom curve. You could explain this simply by saying that g represents a growth in the collecting and so a larger g implies faster growth. Or you could say that when $g = 0$, the system asymptotes at an equilibrium.

(e) Both of these curves initially decrease over time. Prove, however, that the curves will not decrease if $n_0 = 0$.

From the differential equation, when $t = 0$ and $n = 0$, then initially $\frac{dn}{dt} = -f\,n + c + g\,t$ will be nearly $\frac{dn}{dt} = c$. As c is positive (stated when describing the model) then n will exhibit positive growth.

(f) For what value of n_0 would the collections neither shrink nor grow initially when the museum opens (near $t = 0$)?

From the differential equation, when $t = 0$, then $\frac{dn}{dt} = -f\,n + c + g\,t$ will be nearly $\frac{dn}{dt} = -f\,n + c$, which will stay roughly constant at first if the number of specimens is c/f.

(g) **[NOTE: This problem involves a bit of algebra. To help a bit, let's assume *c* = 0 for this part.]** Check that the general solution, $n = e^{-f\,t}\left(\frac{g}{f^2} + n_0\right) + \frac{f\,g\,t - g}{f^2}$, does indeed satisfy the differential equation, $\frac{dn}{dt} = -f\,n + g\,t$, by using the general solution to calculate $\frac{dn}{dt}$ and confirming that the result is the same as the original differential equation [assuming *c* = 0 throughout].

Taking the derivative of the general solution with respect to time we have $\frac{dn}{dt} = \frac{d(e^{-f\,t}\left(\frac{g}{f^2}+n_0\right)+\frac{f\,g\,t-g}{f^2})}{dt} = -fe^{-f\,t}\left(\frac{g}{f^2} + n_0\right) + \frac{f\,g}{f^2}$. At this point, you can use the fact that $n = e^{-f\,t}\left(\frac{g}{f^2} + n_0\right) + \frac{f\,g\,t - g}{f^2}$ to write that $n - \frac{f\,g\,t - g}{f^2} = e^{-f\,t}\left(\frac{g}{f^2} + n_0\right)$. Then, using this to replace $e^{-f\,t}\left(\frac{g}{f^2} + n_0\right)$ in the derivative with $n - \frac{f\,g\,t - g}{f^2}$, we get $\frac{dn}{dt} = -f\left(n - \frac{f\,g\,t - g}{f^2}\right) + \frac{f\,g}{f^2}$. Cleaning up, this equals $\frac{dn}{dt} = -fn + gt$, as required.

OK to get to $\frac{dn}{dt} = -fe^{-f\,t}\left(\frac{g}{f^2} + n_0\right) + \frac{f\,g}{f^2}$ and then note that for this to equal $\frac{dn}{dt} = -f\,n + g\,t$ requires that $-fe^{-f\,t}\left(\frac{g}{f^2} + n_0\right) + \frac{f\,g}{f^2} = -f\,n + g\,t$, rearranging to get n, this will be true if the general solution equals $n = e^{-f\,t}\left(\frac{g}{f^2} + n_0\right) - \frac{g}{f^2} + \frac{g\,t}{f}$, which does equal the general solution as stated in the question.

Question 3 [10 points]: For each of the following *Mathematica* outputs, say which input code could have generated the output [**Circle one**]:

Output:

{{x -> 1}, {x -> 6}}

Input:

(a) Solve[x^2 – 7 x + 6 == 0, x]

(b) Solve[x^2 – 7 x + 6 = 0, x]

(c) Solve[x^2 – 7 x + 6 := 0, x]

(d) Solve[x^2 – 7 x + 6, x]

Output:

1 - 3 x

Input:

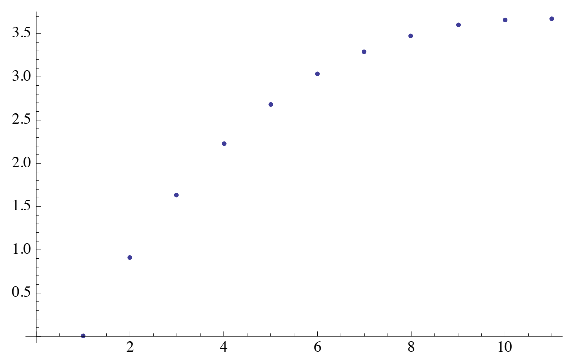
(a) Factor([1 - 9 x^2]/[1 + 3 x])

(b) Factor[(1 - 9 x^2)/(1 + 3 x)]

(c) Factor[1 - 9 x^2/1 + 3 x]

(d) Factor{(1 - 9 x^2)/(1 + 3 x)}

Output:



Input:

(a) ListPlot[x Exp[-0.1 x], {x, 0, 10}]

(b) Plot[x Exp[-a x], {x, 0, 10}]

(c) ListPlot[Table[x Exp[-0.1 x], {x, 0, 10}]]

(d) Plot[x Exp[-0.1 x], {x, 0, 10}]

Output:

r

Input:

(a) Factor[D[r n (1-n K) - h, n]/. n->0]

(b) Factor[D[r n (1-n K) - h, n /. n->0]]

(c) Factor[D[r n[t] (1- n[t] K) - h, n]/. n[t]->0]

(d) Factor[D[r n[t] (1- n[t] K) - h, n /. n[t]->0]]