Biomathematics 301 Midterm 2014

Name_____

Student #_____

Signature_____

Question 1 [15 points]: Wolbachia is a bacteria that frequently infects insects and is passed from mother to offspring (via the cytoplasm). Caspari and Watson (1959) modeled the frequency, p, of individuals infected with Wolbachia, assuming that infected individuals have a lower fitness, $1-s$, but uninfected females who mate with infected males suffer a cytoplasmic incompatibility reaction, reducing their fertility by a factor $1-\sigma$. The continuous-time version of this model is:
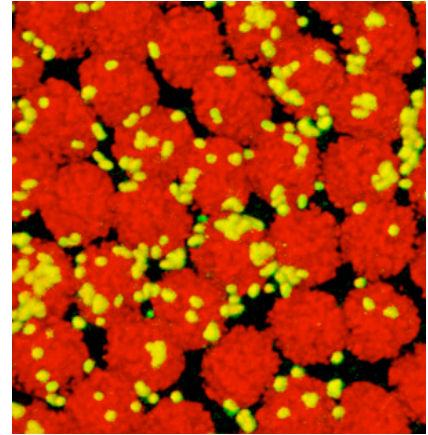

Fruit fly cells (red) infected by Wolbachia (yellow). - *University of Rochester*

$$\frac{dp}{dt} = -s\,p\,(1-p) + \sigma\,p^2\,(1-p)$$

(a) What are the equilibria of this model? **[Show your work]**

(b) The following graph shows the differential equation plotted against the current frequency of infected females for $s = 0.045$ and $\sigma = 0.3$. On this graph, mark all equilibria (X) and specify whether each one is locally stable (LS) or unstable (LU), based on the information contained in the graph.
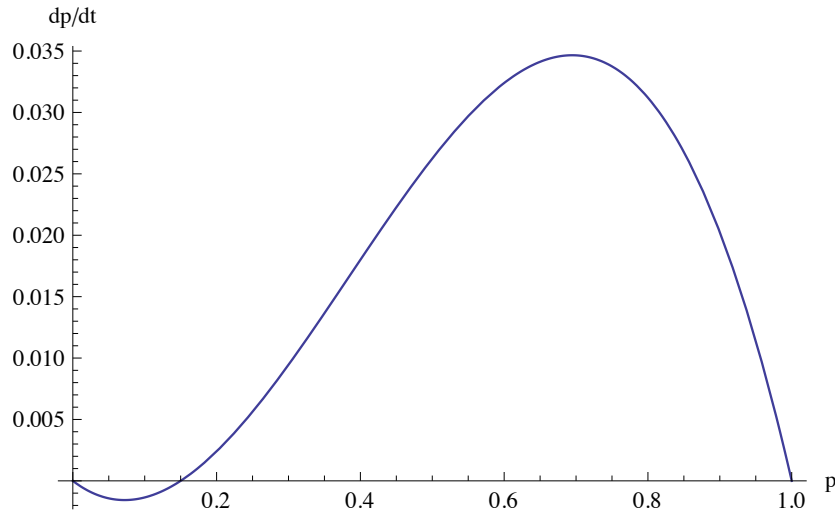
(c) A group is developing a strategy to fight Dengue fever by introducing Wolbachia-infected mosquitoes into the field. This particular strain of Wolbachia causes mosquitoes to die before the Dengue virus reaches their salivary glands, reducing the chance that a person bitten by a mosquito contracts the disease. If $s = 0.045$ and $\sigma = 0.3$ (as in the above figure), what advice would you give the researchers about the fraction of infected mosquitoes to introduce, given that they want the Wolbachia infection to spread in order to reduce the transmission of Dengue fever? [**Make a quantitative prediction using the information provided.**]

Question 2 [15 points]: Hand-washing is a key step in the prevention of hospital-acquired infections, yet the compliance levels are surprisingly low and variable, with baseline rates ranging from 5% to 89% and an overall average of 38.7% according to a 2009 WHO study[1]. Peer pressure and the norms of a hospital play an important role in the practices of a doctor. Let $p(t)$ represent the fraction of doctors at time t that follow appropriate hand-washing routines ("compliant"). Assume that, in the absence of peer pressure, a fraction α of the non-compliant doctors become compliant per time step. This fraction goes up by an additional $\beta p(t)$ in a particular hospital because compliant doctors are encouraged to ask non-compliant doctors to wash their hands. Also assume that a fraction, f, of doctors that are compliant stop following hand-washing procedures per time step.

(a) Complete the following flow diagram for this hospital. **Add arrows** and write in **total flow rates** associated with each arrow (i.e., if a flow rate is per capita, multiply it by the appropriate variable).
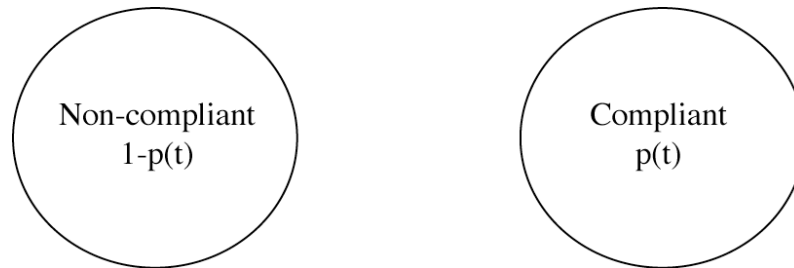
(b) Write down the recursion equation for the fraction of compliant doctors, $p(t)$, at time t. Assume that the time step is small (e.g., by day), so that doctors don't switch more than once within a time step.

[1] http://whqlibdoc.who.int/hq/2009/WHO_IER_PSP_2009.07_eng.pdf

Question 3 [30 points]: In class, we modeled diploid selection in discrete time. Here we consider the continuous time version. Let "recursion" stand for the equation for $p(t+1)$ in the discrete-time diploid selection model and measure the relative fitnesses using $W_{AA} = 1+s$, $W_{Aa} = 1+hs$, and $W_{aa} = 1$, where s is the selection coefficient favoring allele A and h is its dominance coefficient.

(a) I used one of the following *Mathematica* codes and the definition of a derivative to obtain a differential equation for the diploid model. For each case, say whether the method is correct or incorrect. If incorrect, diagnose the problem.

Code	Correct? [Yes or No]	Diagnosis
D[recursion, p] **% /. s -> s*Δt** **%/Δt** **Limit[%, Δt -> 0]**		
Factor[recursion - p] **% /. s -> s*Δt** **%/Δt** **Limit[%, Δt -> 0]**		
Factor[recursion - p] **% /. p -> p*Δt** **%/Δt** **Limit[%, Δt -> 0]**		

(b) The resulting differential equation is $\dfrac{dp}{dt} = s\,p\,(1-p)(h+p-2\,h\,p)$. Begin a separation of variables by rewriting the differential equation with the variables separated:

$$\int \qquad\qquad\qquad\qquad = \int$$

[Do not carry out the integration!]

4

(c) The equilibria of this model occur at $\hat{p} = 0$, $\hat{p} = 1$ and $\hat{p} = h/(2h-1)$. Determine the values of h for which this last equilibrium is biologically valid (i.e., $\hat{p}$ is a frequency). Make no assumption about the sign of h. [**Hint:** It may help to handle cases where $(2h-1)$ is positive and negative separately.]

(d) Perform a stability analysis for the model of diploid selection, $\dfrac{dp}{dt} = s\,p\,(1-p)(h+p-2\,h\,p)$, at the equilibrium $\hat{p} = 0$. What is the condition required for this equilibrium to be stable?

(e) Fill in the blanks below to complete the Mathematica code that you would use to perform the stability analysis for the equilibrium $\hat{p} = h/(2h-1)$:

D[s p (1 - p) (h + p - 2 h p) , _____]

Factor[%_____]

(f) *Mathematica* answers $(1-h)h\,s/(2h-1)$, which can be simplified to $(1-h)\hat{p}\,s$. Based on this information, are the following statements true or false?

True or False [Circle one]: The approach to $\hat{p} = h/(2h-1)$ will be oscillatory when $h > 1$.

True or False [Circle one]: $\hat{p} = h/(2h-1)$ is locally stable when $h > 1$.

Question 4 [30 points]: The extinction of a species can cause a cascade of extinction events for species that depend on it. Consider a plant that is exponentially declining at rate r (e.g., due to habitat loss), with the number of plants at time t given by the general solution for the exponential model, $p(t) = p_0\,e^{-rt}$. Here we model the rate of decline of an herbivore, $h(t)$, that depends fully on this plant for food according to the differential equation:

$$\frac{dh}{dt} = f\,p\,h - d\,h$$
$$= f\left(p_0\,e^{-rt}\right)h - d\,h$$

where f is the rate of consumption of the plant by the herbivore and d is the death rate of the plant.

(a) What is the general solution for $h(t)$ in this model, assuming an initial number h_0 of herbivores?

(b) Write down one check that you could perform to make sure that your general solution is reasonable and say whether or not your solution passes or fails the check:

Check:_____**Passes or Fails**

_____[Circle one]

(d) The following plot shows this model for a particular community:
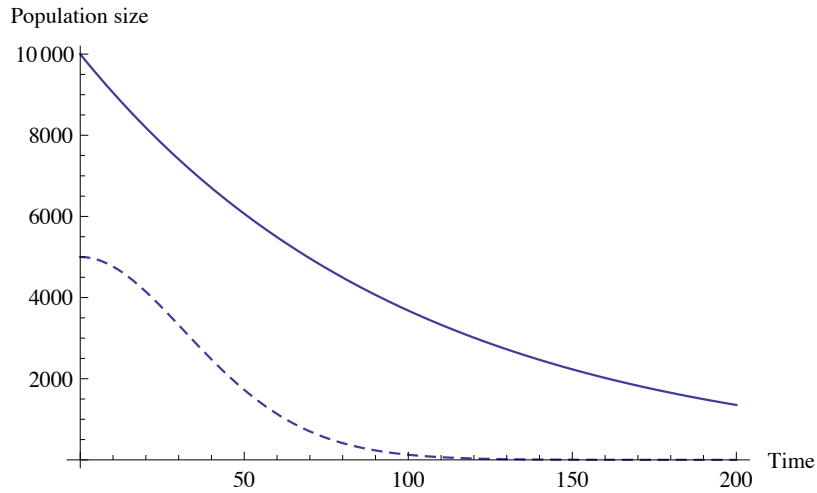
Population size

Circle one: The plant population is shown as a curve that is **Solid or Dashed**.

Circle one: The herbivore population is shown as a curve that is **Solid or Dashed**.

Justify your choice in one or two sentences.

_____

_____

_____

(e) The plant grows in two communities. In the first community, the plant's herbivore is a small beetle that eats a small amount of leaves. In the second community, the plant's herbivore is a mammal that eats whole plants.

Circle one: The model analyzed here is more appropriate for the **First or Second** community.

Justify your choice in one or two sentences.

_____

_____

_____