Bio 301 Review Problems

(Additional review problems with answers are available on the course web page)

Probability Question: Foxes

(a) If foxes are randomly found over the habitat and move independently of one another, the chance of observing *k* foxes in a given area is Poisson distributed. Using a mean number of foxes per km^2 of 10, the chance of observing two or fewer foxes is:

$$\sum_{k=0}^{2} \frac{e^{-\mu}\mu^{k}}{k!} = e^{-10} + 10e^{-10} + 50e^{-10} = 0.0028$$

(b) For a Poisson distribution, the variance equals the mean, which equals 10. The standard deviation of the Poisson is the square-root of the variance, or 3.16. Thus, observing 20 foxes is approximately (20-10)/3.16 = 3.16 standard deviations above the mean of ten, which is considered to be quite significant (remember that approximate 95% confidence limits for the Normal Distribution are +/- 2SD).

(c) The results of sampling in the two years might have differed because the fox density naturally varies over time, so that 10 foxes/km^2 is not the expected value for each year, but only an average of the expected values over many years. Alternatively, the distribution of foxes over space might not be independent but be clumped, making it more likely to observe either many fewer or many more foxes within a sample.

Probability Question: Breast Cancer

(a) The chance of getting breast cancer in the next ten years without a mastectomy is?

Using an exponential: $$\int_{t=0}^{10} \lambda e^{-\lambda t} dt = \Big|_{t=0}^{10} - e^{-\lambda t} = -e^{-0.05(10)} + e^{-0.05(0)} = 0.393$$

Using a geometric: $$\sum_{x=1}^{10} p(1-p)^{x-1} = \sum_{x=1}^{10} 0.05\,(1-0.05)^{x-1} = 0.401$$

Using a Poisson: $$1-\sum_{k=0}^{0}\frac{e^{-\mu}\mu^{k}}{k!}=1-\frac{e^{-10*0.05}(10*0.05)^{0}}{0!}=0.393$$

Using a binomial: $$1-\binom{10}{0}0.05^{0}\,(1-0.05)^{10}=0.401$$

(b) The chance of getting breast cancer in the next ten years with a mastectomy is?

Using an exponential: $$\int_{t=0}^{10}\lambda e^{-\lambda t}dt=\Big|_{t=0}^{10}-e^{-\lambda t}=-e^{-0.005\,(10)}+e^{-0.005\,(0)}=0.0488$$

Using a geometric: $$\sum_{x=1}^{10}p(1-p)^{x-1}=\sum_{x=1}^{10}0.005\,(1-0.005)^{x-1}=0.0489$$

Using a Poisson: $$1-\sum_{k=0}^{0}\frac{e^{-\mu}\mu^{k}}{k!}=1-\frac{e^{-10*0.005}(10*0.005)^{0}}{0!}=0.0488$$

Using a binomial: $$1-\binom{10}{0}0.005^{0}\,(1-0.005)^{10}=0.0489$$

(c) Based on distributions calculating the waiting time, you could use an exponential or a geometric distribution. The exponential assumes that the risk data provided by the doctor is an instantaneous rate, whereas the geometric assumes that the risk data is the total risk of breast cancer over a year. Note that, if 0.05 represents the instantaneous risk of cancer, then the chance of getting cancer within one year is (from the exponential distribution, given cancer appears at a rate 0.05 and we watch for an interval of one year):

$$\int_{t=0}^{1}\lambda e^{-\lambda t}dt=\Big|_{t=0}^{1}-e^{-\lambda t}=-e^{-0.05\,(1)}+e^{-0.05\,(0)}=0.0488$$

which is less than 0.05. This reflects the fact that, as time passes, it is more and more likely that an individual will have already gotten breast cancer, which reduces the future chances of getting breast cancer for the first time.

Alternatively, we can think about this as a problem about how many events happen in ten years, where there must be ≥1 event for her to develop cancer. To calculate this, it is easier to calculate the probability that she does not contract cancer in any year and subtract this from one. In continuous time, this would be a Poisson distribution (observation over an interval of ten years). Considering each year discretely, there would be ten years during which an event could occur, and the chance that it doesn't occur in any year would be binomial.

The patient decides that a 40% chance of breast cancer within ten years is too high and decides to have the mastectomy.

Probability Question: Heart attack drug

(a) Answer this question using a Binomial Distribution.

$$\Pr(X = 0 \text{ heartattacks}) = \binom{100}{0} p^0 (1-p)^{100} = (1-0.03)^{100} = 0.0475$$

(b) Answer this question using a Poisson Distribution.

$$\Pr(X = 0 \text{ heartattacks}) = \sum_{k=0}^{0} \frac{e^{-\mu}\mu^k}{k!} = e^{-\mu} = e^{-(100)(0.03)} = 0.0498$$

(c) The answer based on the Binomial should be more accurate. It must be the case that the total number of heart-attack patients receiving the drug will lie between 0 and 100 (remember that the Poisson is unbounded), and it is reasonable to assume that the chance that each individual suffers a heart attack would be independent. The Poisson provides an adequate approximation in this case, however, because n is large (100) and p is small (0.03).

Either way, you decide that your drug is probably effective (since there is less than a 1/20 chance of seeing no heart attacks after one year). But just to be sure, you extend the trial for another six months.

Linear Model with Multiple Variables: Seed Germination

(b) Write a transition matrix for the number of seeds of each type on day t+1 as a function of the number of seeds of each type on day t.

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 \\ \alpha & 1-\beta \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

(c) What are the eigenvalues for this matrix? $1 - \alpha$ and $1 - \beta$, because the transition matrix is diagonal.

(d) What are the eigenvectors for this matrix?

$\{1, \alpha/(\beta-\alpha)\}$ associated with the eigenvalue $1 - \alpha$

$\{0,1\}$ associated with the eigenvalue $1 - \beta$

(e) Write down the matrices, **D**, **A**, and $\mathbf{A}^{-1}$ that can be used to obtain a general solution for the model.

$$\mathbf{D} = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ \dfrac{\alpha}{\beta-\alpha} & 1 \end{pmatrix} \qquad \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{\alpha}{\beta-\alpha} & 1 \end{pmatrix}$$

(f) Write down the exact general solution for the number of dormant and activated seeds.

$$\begin{aligned} \mathbf{A}\mathbf{D}^t\mathbf{A}^{-1} &= \begin{pmatrix} 1 & 0 \\ \dfrac{\alpha}{\beta-\alpha} & 1 \end{pmatrix} \begin{pmatrix} (1-\alpha)^t & 0 \\ 0 & (1-\beta)^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{\alpha}{\beta-\alpha} & 1 \end{pmatrix} \\ &= \begin{pmatrix} (1-\alpha)^t & 0 \\ (1-\alpha)^t \dfrac{\alpha}{\beta-\alpha} - (1-\beta)^t \dfrac{\alpha}{\beta-\alpha} & (1-\beta)^t \end{pmatrix} \end{aligned}$$

Giving us the general solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (1-\alpha)^t & 0 \\ (1-\alpha)^t \dfrac{\alpha}{\beta-\alpha} - (1-\beta)^t \dfrac{\alpha}{\beta-\alpha} & (1-\beta)^t \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

The number of dormant seeds thus changes according to:

$$x(t) = (1-\alpha)^t x(0)$$

and the number of activated seeds according to:

$$y(t) = \left((1-\alpha)^t \frac{\alpha}{\beta-\alpha} - (1-\beta)^t \frac{\alpha}{\beta-\alpha}\right)x(0) + (1-\beta)^t y(0)$$

(g) On the basis of part (f), what is the equilibrium number of dormant and activated seeds (make a reasonable assumption about the values of α and β).

Assuming that α and β are fractions between zero and one, $(1-\alpha)^t$ and $(1-\beta)^t$ will decay to zero. So the equilibrium towards which the system tends is zero dormant seeds and zero activated seeds. This makes sense, because there are no new seeds added to the system in this model.

(h) If germination of activated seeds is very rapid (β>>α), then after a few days, what proportion of the seeds that remain would be dormant?

If β>>α, then the largest eigenvalue in magnitude will be 1 – α. The eigenvector associated with this eigenvalue is {1, α/(β–α)}, which indicates that the fraction of dormant seeds will tend towards 1/(1 + α/(β–α)) = (β–α)/β. This answer can also be obtained from the general solution by letting $(1-\beta)^t$ tend to zero and calculating $x(t)/(x(t)+y(t))$.

Linear Model with Multiple Variables: Stage-structured population

Consider an organism that has two size classes: small in the first year and large in the second year. Small individuals either die by the next census $(d_1 = 2/3)$ or survive to become large individuals ($p_1 = 1/3$). Large individuals never live for more than one year.

You have been censusing this poplation yearly and have found that the small individuals produce one offspring on average every year that survive until the next census ($m_0 = 1$). The large individuals are much more fertile and produce six offspring on average that survive until the next census ($m_1 = 6$).

(a) Write down the Leslie matrix for this population.

$$\mathbf{L} = \begin{pmatrix} 1 & 6 \\ 1/3 & 0 \end{pmatrix}$$

(b) What is the long-term rate of growth of this population?

The two eigenvalues of **L** are 2 and –1, so that the long-term growth rate is 2 (doubling each year).

(c) In the long-term, what proportion of the population is expected to be large?

The eigenvector associated with the eigenvalue of 2 is $\{1, 1/6\}$, which indicates that the fraction in the large (second) stage class is 1/6/(1+1/6) = 1/7.

(d) Would you expect it to take a long time or a short time for the proportion of large individuals to reach the value in (c)? Justify your answer.

It should take a short time, because the second eigenvalue is much smaller than the first. That is, 2^t will rapidly outstrip 1^t in importance as time increases.

Non-Linear Model with Multiple Variables: The Predator-Prey Model

(a) Determine the two equilibria of these equations. [CAREFUL: Double check that the numbers of *both* predators and prey do not change over time when started at an equilibrium.]

Equilibrium 1: $H(t) = P(t) = 0$ (predators and prey absent)

Equilibrium 2: $H(t) = \delta/c$ and $P(t) = r/\beta$ (predators and prey present)

(b) Determine the local stability matrix that approximates these equations near the equilibrium with both species absent. Repeat, finding the local stability matrix near the equilibrium with both species present.

Equilibrium 1: $\mathbf{M} = \begin{pmatrix} r & 0 \\ 0 & -\delta \end{pmatrix}$

Equilibrium 2: $\mathbf{M} = \begin{pmatrix} 0 & -\beta\delta / c \\ cr / \beta & 0 \end{pmatrix}$

(c) Find the eigenvalues for the two matrices in Part 2.

Equilibrium 1: r and $-\delta$

Equilibrium 2: $\lambda = \pm\sqrt{-r\delta}$

(d) From these eigenvalues, determine whether each equilibrium is stable or unstable, assuming that every parameter is positive. [NOTE: An equilibrium in a continuous-time model is stable if the real part of the eigenvalues are negative. The real part of a complex eigenvalue $a + b\,i$ is a.]

Equilibrium 1: Unstable because r is positive.

Equilibrium 2: Cycling because the part in the square-root term is negative, without moving rapidly in or out, because the real part of the eigenvalue is zero. To this order, the equilibrium is neutrally stable, but we would need to do more than a linear stability analysis to determine whether the system might slowly move in or out.

(e) Compare your answers from (a) – (d) here to Homework 9 based on the discrete-time model.