Supplementary Materials for

Exit time as a measure of ecological resilience

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This PDF file includes:

Other supplementary material for this manuscript includes:

Movies S1 and S2
MATLAB code (zipped folder)

Methods

Example data sets

Phycocyanin in Lake Mendota

This data set consists of concentrations of phycocyanin in Lake Mendota, Wisconsin (*44*) in the year 2011, and is available at https://lter.limnology.wisc.edu/data. The phycocyanin concentrations are recorded every minute throughout the year while the lake surface is ice-free. We used standardized values of phycocyanin (z-score of $\log_{10}$(phycocyanin RFU) see (*44*)). For our example we analyzed a period during thermal stratification when Cyanobacteria blooms are common (*44*). The data resolution is 1 minute. In the reconstructions we used 50 bins and τ values ranging from 1 to 10 steps. In stochastic simulations we use a fixed time step of 0.01.

Climate data set

This data set spans from the end of Eemian interglacial to the beginning of the current Holocene interglacial where the climate alternated between cold glacial and warmer interstadial. The calcium record from the GRIP (Greenland Ice Core Project) has the highest temporal resolution (*48*), being annual and covers most of the last glaciation and DO events among ice-core records (*49*). The data include a few low values that have a visible effect on the reconstructed model, but we decided not to filter those out for the current analysis. In the reconstructions, we used 50 bins and a τ values ranging from 1 to 12 steps. In stochastic simulations we use a fixed time step of 0.001.

The Langevin approach

The method called 'Langevin approach' (*36-40*) is used to estimate the functions of drift $D_1(x)$ and diffusion $D_2(x)$. Here we give an overview of the method, for computational details we refer to the original references (*36-40*) and (*79*).

Step 1: Check if the data is suitable

Before we can apply the method, we need to check if the data set fulfils the assumptions for the method. The three main conditions are (*38*):

(1) The time series should have a high resolution and be sufficiently long (see next section).

(2) The data should be stationary which means that the statistical properties of data remain constant over time. Note that if this is violated, we can still apply the Langevin method to shorter (overlapping) time windows (*38*).

(3) The noise source of the Langevin equation should be uncorrelated (white). This implies that the data should be Markovian, i.e. the future state should depend on the current state solely.

If part of the conditions are not fulfilled this does not need to be damning (*38*). For instance, if data are not Markovian one can find a coarser time scale called 'Markov-Einstein' time scale (*64*) and apply the analysis to the corresponding coarser dataset.
Upon visual inspection of our data sets (Figures 4A,5A), it seems reasonable to assume that they are stationary. Additionally, we tested this using the augmented Dickey-Fuller test (*80*) (MATLAB function `adftest`, econometrics toolbox) using an autoregressive model with drift. The test confirmed that both datasets are stationary (Lake Mendota: D-F statistic= -10.2 , $p<0.001$, Climate: D-F statistic=-6.4, $p<0.001$).

We determined this Markov-Einstein time scale using a method (*81*,*82*) that is well suited for relatively small data sets. This method validates the Chapman-Kolmogorov equation that describes how a conditional probability density function $p(x_3, t_3|x_1, t_1)$ in a Markov process can be computed from two smaller steps using an intermediate time t_2 ($t_1<t_2<t_3$):

$$p(x_3, t_3|x_1, t_1) = \int p(x_3, t_3|x_2, t_2)\, p(x_2, t_2|x_1, t_1)\, dx_2 \qquad \text{(S1)}$$

This is a necessary but strong condition for a Markov process and can also be considered as a sufficient condition for the Markov property if it is valid for all sampling times (*81*). The Markov-Einstein time scale is determined as the time interval where a χ^2 distance (see equation 16.12 in *82*) between the conditional probability density function $p(x_3, t_3|x_1, t_1)$ estimated directly from the data and determined by the right-hand side of equation S1 is minimal. For different time lags, we consider the intermediate times to be in the middle of the interval [t1, t3]. For details we refer to (*82*).

Visual inspection of the results for our data examples suggested a good agreement between the distributions determined directly or with the Chapman-Kolmogorov equation for nine evenly spaced values of x_1 (gray bars in Figure 6E, F). Furthermore, we found an indication that the Markov-Einstein time scale for the climate data was very small (Figure 6H). For the cyanobacterial data there was a minimum at lag 5, but the differences between lag 1-5 were very small (Figure 6G). As we found no evidence that the data sets were non-Markovian, we do not need to reduce the resolution. Furthermore reducing the data resolution not only would make the datasets very short, but can also cause larger systematic errors termed 'finite-time effects' (*83*). It has for example been shown that irrespective of the true model, the drift will always be linear and the diffusion always quadratic, if the data resolution is too coarse (*83*). Therefore, for near-Markovian data sets it is not recommended to reduce data resolution (*62*).

Step 2: Perform the analysis

For the actual reconstruction process the data should be divided into as many bins as needed to describe the functions. In each of these bins we determine the drift by averaging the estimated rates of change. The diffusion is also determined by estimating the variance of the rates of change estimated for each bin.

In mathematical terms, for the n-th Kramers-Moyal coefficient D_n, we need to calculate the n-th conditional moment for each bin:

$$M_n(x,\tau) = E((x(t+\tau) - x(t))^n | x(t) = x), \quad n = 1,2,4 \tag{S2}$$

Where x is the bin center and τ is the sampling time between system states at times t and $t+\tau$. We estimated these conditional moments $M_n(x,\tau)$ using the nonparametric Nadaraya-Watson estimator (*84*) (see also (*85*) or (*86*) for details). Subsequently each of the coefficients D_1 and D_2 can be calculated as:

$$D_n(x) = \frac{1}{n!} \lim_{\tau \to 0} \frac{M_n(x,\tau)}{\tau}, n = 1,2,4 \tag{S3}$$

We further need the fourth coefficient D_4 for the validation of the method. To estimate D_n (equation S3) for each bin, we calculate conditional moments $M_n(x,\tau)$ for the first few time lags τ, fit a regression line to $M_n(x,\tau)$ versus τ data, and finally take the slope as the estimate for $D_n(x)$. Typically, only a few multiples of sampling time Δt are considered for these regressions (we took $\tau = \Delta t, \dots, 10\Delta t$ for the phycocyanin dataset and $\tau = \Delta t, \dots, 12\Delta t$ for the climate dataset. See validation). For a successful reconstruction, the sampling intervals of the data should be much smaller than the relaxation time of the system. One can roughly estimate the relaxation time of the yet (unknown) data generating system by fitting an exponential exp(-c t) to the autocorrelation function of data for the first few lags and then take 1/c as a rough estimate for the relaxation time (*83*). We estimated these relaxation times to be ca. 180 lags for the climate data, and ca. 680 lags for the phycocyanin data. Accounting for the finite-τ corrections for the diffusion coefficient, one can estimate the drift and diffusion coefficients by finding the slope of a weighted linear regression as (*38*):

$$\begin{aligned} M_1(x,\tau) &\approx D_1(x)\tau \\ M_2(x,\tau) &\approx 2D_2(x)\tau + (D_1(x)\tau)^2 \end{aligned} \tag{S4}$$

The statistical uncertainties of M_1 and M_2 for each bin x and lag τ can be directly estimated from data (*87*)

$$\sigma^2{}_{M_1}(x,\tau) = \frac{M_2(x,\tau) - {M_1}^2(x,\tau)}{N_x}, \sigma^2{}_{M_2}(x,\tau) = \frac{M_4(x,\tau) - {M_2}^2(x,\tau)}{N_x} \tag{S5}$$

Where N_x is the number of data in bin x. The variances of drift and diffusion coefficients for a fixed state x, i.e., $\sigma^2{}_{D1}(x)$ and $\sigma^2{}_{D2}(x)$, then follow by finding the uncertainties of the slopes of the mentioned weighted linear regression lines (equation S4) with $\sigma^2{}_{M1}(x,\tau)$ and $\sigma^2{}_{M2}(x,\tau)$ as the reciprocals of the weights for the considered time lags τ. However, we did not use equation S5 to calculate the uncertainties of drift and diffusion functions, since these uncertainties do not directly tell us the uncertainties of exit time curves. Instead we followed an alternative Monte-Carlo method to estimate such uncertainties (see the section "Monte-Carlo error propagation approach"). Afterwards, we fitted weighted smoothing splines to the drift and diffusion functions being estimated at each bin x with $\sigma^2{}_{D1}(x)$ and $\sigma^2{}_{D2}(x)$ as the reciprocals of the corresponding weights.
Using the reconstructed model, we determine the mean exit time applying equation S15. For simplicity, we used the drift function to define the borders of the basin of attraction, ignoring the effects of multiplicative noise (see Glossary: *basin of attraction*).

Step 3: Validate the model

According to the Pawula Theorem (*38*, *88*), the fourth coefficient D_4 should be close to zero if the modelled process is well described by a drift D_1 and diffusion D_2 coefficients only. Otherwise, the process under study has a much more complex dynamics and requires infinitely many D_n coefficients to be described. As a criterion, D_4 should be small compared to the diffusion coefficient D_2. In order to make non-dimensional comparison possible we express the state in units of the standard deviation of the whole data set (*89*). For both data sets, D_4 is much smaller than D_2 (Figure 6A,B).

In our experience, the time scale of the reconstructed model can be biased if the data resolution is not optimal. As this is essential for derivation of the exit time, we recommend checking whether data sets generated with the fitted models and the corresponding data have the same statistical properties. For this purpose, we used the one-step conditional probability distribution (*40*, *64*), which is the probability distribution of the next state given the present state for each bin of x_0, i.e., $p(x_1|x_0)$. We compare these conditional distributions determined with generated data from the fitted model with the distributions obtained from the original data (Figure 6C,D). For instance, if the diffusion coefficients, D_2 would be modelled inaccurately those distributions would not match well. In both our data examples, it can be seen that the probability distribution around each starting bin is quite similar in the model-generated and the original data, suggesting that the modeled diffusion captures the local stochasticity well.

For the sake of comparison between the datasets and their corresponding reconstructed models we also plot an example simulation for each dataset (Figure 6I,J with the same number of data points for each dataset as the original data sets (Figure 4A and 5A).

As we are specifically interested in the mean exit time, we also compare the estimated mean exit time with the mean exit time derived directly from the data sets using a straightforward survival time analysis (*61*). We use here a simple definition for mean exit time: namely the time it takes to reach the deterministic basin boundary starting from each of the stable states. This was analyzed by determining the time between crossings of the stable and unstable states in each of data sets. We did the same analysis with 1000 generated data sets from the reconstructed model with the same length and resolution as the original data. Although there is not a perfect fit (Figure 7), even not for the modelled data set (Figure 7A,D), the distributions of exit times have a similar distribution.

Monte-Carlo error propagation approach

Based on the approach of (*62*), we designed a Monte-Carlo error propagation procedure to estimate uncertainty of exit time from particular specified basin of attraction. The method takes advantage of the stochastic Langevin model that is estimated based on the original time series. This stochastic "parent model" is run repeatedly by using the Euler Maruyama solver (time step 0.01 (Mendota) or 0.001 (Climate and overgrazing model)) to create 1000 pseudo data sets with the same length and resolution as the original data set. The initial conditions are drawn from the stationary probability density function of the parent model.

With each of the pseudo data sets we reconstruct new Langevin equations. Per bin of the drift and diffusion functions (D_1 and D_2) we adjust for bias by comparing the medians per bin with the binned values of the parent model. The mean exit time of each model is then determined with the bias adjusted drift D_1 and diffusion D_2 functions. For the mean exit times we use the same domains as derived from the parent model.

Sensitivity analysis

Using the May model (main text equation 4) with default parameters (listed in Figure 1) we analyzed the effect of the quality of the data set and some parameters of the method on the uncertainty in the mean exit time and the bias of the model.

For each tested setting, we performed a Monte-Carlo error propagation procedure (see above), with 100 replicates and the May model (main text equation 4) as parent model.

Glossary including equations

This glossary is meant to give a simplified didactical overview of the theory. For the ease of explanation, we confine ourselves to one-dimensional systems. For the full details, mathematical

proofs and more general formula we refer to textbooks of stochastic processes, we recommend in particular (*26*) and (*88*).

Italic terms are explained elsewhere.

Absorbing boundary conditions

If a stochastically moving particle hits an absorbing barrier it is removed from the system, i.e. the barrier absorbs. In partial differential equations an analogous behavior is achieved for any state by setting a boundary condition where the dependent variable (e.g., probability density $P(x,t)$ (equation S19) or survival $S(x_0,t)$ (equation S22)) is constrained to zero. We use these boundary conditions for instance in the *backward* and *forward Fokker Planck equation* at the unstable edge between two basins of attraction, where a small perturbation can cause a regime shift between basins. At this boundary the mean exit time is zero: $T(x_0) = 0$.

Additive noise

A *Langevin equation* is called additive if the strength of noise $\sigma(x)$ does not depend the state variable:

$$\sigma(x) = \sigma \tag{S6}$$

Backward Fokker-Planck equation

The Backward Fokker-Planck equation is a partial differential equation that can be used to calculate the time that a system will stay in a certain interval of states. It is called the backward equation as it describes the evolution of a conditional probability distribution $p(x,t|y,s)$ with respect to the 'backward' variables (y,s) (*26*, *90*).

This equation is applied to calculate the *survival probability function* and *mean exit time*.

Basin of attraction

Largest range of initial states of a state variable where a deterministic model goes to an equilibrium. For a *Langevin equation* this range is less well defined due to the stochasticity. The position of equilibria can shift due to *multiplicative noise (79)*. In our systems, this effect was minor and for simplicity we used the basin of attraction of the *deterministic part* of the model, where unstable equilibria (i.e. solutions of $f(x)=0$ with a positive slope) define the basin borders. To account for the effects of *multiplicative noise*, you could alternatively use local maxima of the *effective potential* as basin borders (equation S12).

Deterministic part of Langevin equation f(x)

This is the deterministic part of the *Langevin equation.* It is equal to the *drift term* (D_1) of the *forward or backward Fokker-Planck equations.*

Diffusion term ($D_2(x)$)

Used in the *forward* or *backward Fokker-Planck equations* to describe stochasticity. This term can be expressed in terms of the *stochastic part* of the *Langevin equation* $\sigma(x)$:

$$D_2(x) = \frac{1}{2}\sigma(x)^2 \quad \text{(S7)}$$

Distribution of exit time

The *mean exit time* $T(x_0)$is the average time it takes for trajectories of the *Langevin equation* starting at an initial state x_0 to exit a *basin of attraction*. The cumulative distribution function $CDF_T(x_0, t)$ of exit times follows directly from the *survival probability function* $S(\mathrm{x_0}, \mathrm{t})$ (see equations S21-S23):

$$CDF_T(x_0, t) = 1 - S(x_0, t) \quad \text{(S8)}$$

The derivative of this cumulative distribution with respect to t, determines the probability density function of *exit times* $PDF_T(x_0, t)$:

$$PDF_T(x_0, t) = -S'(x_0, t) \quad \text{(S9)}$$

Drift term ($D_1(x)$)

The deterministic part of the *Forward* or *Backward Fokker-Planck equations*. It equals the *deterministic part f(x)* of the Langevin equation:

$$D_1(x) \equiv f(x) \quad \text{(S10)}$$

Effective Potential $U_{eff}(x)$

Stability of one-dimensional dynamic systems can be illustrated by potential functions or 'stability landscape' where valleys and hilltops represent the stable and unstable equilibria respectively. If noise is *additive*, the potential is directly related to the *stationary probability distribution* (equation S18). The effective potential is defined as the potential that includes the effects of *multiplicative noise* on the stationary distribution:

$$p_{st}(x) = exp\left(-U_{eff}(x)\right) \quad \text{(S11)}$$

From the equation of the $p_{st}(x)$ (equation S20) we find a formula for the effective potential in terms of drift and diffusion functions:

$$U_{eff}(x) = logD_2(x) - \int^x \frac{D_1(y)}{D_2(y)} dy \quad \text{(S12)}$$

Where y is a dummy variable and *x* is the state variable. The integral has an arbitrary lower limit. D_1 is the *drift term* and D_2 the *diffusion term*.

Forward Fokker-Planck equation

Partial differential equation that can be used to describe how an initial probability distribution of a certain Langevin system will evolve in time. It is called the forward equation as it describes the evolution of the conditional probability density $p(x, t|y, s)$ with respect to forward variables (x, t) (*26*, *90*). See: *Probability Distribution P(x,t) of the Langevin equation*

Fokker-Planck equation

See *Forward Fokker-Planck equation*.

Half-life

See *median exit time*.

Langevin equation

The Langevin equation is the basic *stochastic differential equation* that describes the dynamics of a state variable *x*:

$$dx = f(x)dt + \sigma(x)dW \quad \text{(S13)}$$

Where *f(x)* is the deterministic part of the equation, dW is the noise source where W is the Wiener process (see also equation S25) and, the function $\sigma(x)$ reflects the intensity of the noise. We interpret the stochastic integral using Itô calculus (see (26)). This model describes single stochastic realizations of the system, whereas the equivalent Fokker Planck equations describe the change deterministically as it is described in probabilistic terms.

Langevin approach

The Langevin approach (*38*) is a set of techniques used to estimate the deterministic $f(x)$ and stochastic parts $\sigma(x)$of the Langevin equation. The method uses no prior assumptions about the shape of functions.

Markov property of a time series

A stochastic process, that is sampled at times t_k as x_k $(k = 1,2,3, \dots, n)$, is referred to be Markovian if

$$p(x_k, t_k | x_{k-1}, t_{k-1}, \dots, x_1, t_1) = p(x_k, t_k | x_{k-1}, t_{k-1}) \quad \text{(S14)}$$

So, the probability of finding a certain state x_k at time t_k conditioned on the full history should be the same as the probability conditioned on the previous state only. This implies that a Markov process depends on the previous state only and have minimal memory. The Langevin approach requires that the time series is approximately Markovian.

Markov-Einstein time scale

The minimum time scale of a timeseries where the process can be considered to be *Markovian*, i.e. the current state is fully determined by the previous state (*82*). Note that this concept is something different than the correlation time scale, which describes the decay of the autocorrelation function (*82*).

Marble-in-a-cup diagram

See: *Potential function.*

Mean exit time $T(x_0)$

The average time it takes for trajectories starting at an initial state x_0 to exit a basin of attraction. From the *backward Fokker-Planck equation*, the following formula can be derived *(26)* to calculate the mean exit time $T(x_0)$:

$$D_1(x_0)\frac{\partial T(x_0)}{\partial x_0} + D_2(x_0)\frac{\partial^2 T(x_0)}{\partial {x_0}^2} = -1 \quad \text{(S15)}$$

Where D_1 is the *drift term* and D_2 the *diffusion term*. Before we can solve this boundary value problem for each basin of the Langevin equation, we have to define the proper boundary values. We should use *absorbing boundaries* for the unstable edge between two basins of attraction and reflecting conditions at the left and right boundaries to indicate that no shift occurs. The choice of location of those reflecting boundaries is somewhat arbitrary, but they should typically enclose the full range of data.

To solve the boundary value problem (equation S15), we used the MATLAB function `bvp4c` or Chebfun (35). The resulting mean exit time $T(x_0)$ refers to a single initial value, to evaluate the mean exit time of the whole *basin of attraction*, we use the *weighted mean exit time* (see equation S24).

Median exit time $T_{med}(x_0)$

This is the time $T_{med}(x_0)$ in which 50% of the trajectories starting at the initial state x_0 will shift to the alternative *basin of attraction*. It can be calculated by determining the time where the survival *probability function* equals 0.5, i.e.:

$$S(\mathrm{T_{med}}(x_0)|x_0) = 0.5 \qquad \text{(S16)}$$

Since the distribution of exit time is often skewed the median exit time can be a more robust resilience indicator than the mean exit time, which could not exist for extremely skewed distributions. The *survival probability function* $S(t|x_0)$ is a solution of equations S21-S23.

Multiplicative noise

The noise of *the Langevin equation* is called multiplicative if its intensity depends on the state, so $\sigma = \sigma(x)$.

Potential function U(x)

Stability of one-dimensional dynamic systems can be illustrated by potential functions or 'stability landscape' that illustrate stable points as valleys and unstable points as hilltops. For deterministic systems, the potential function *U(x)* is the integral

$$U(x) = -\int^{x} D_1(y)dy \qquad \text{(S17)}$$

where *y* is a dummy variable, the integral has an arbitrary lower limit and $D_1(y)$ is the *drift term* of the *Langevin equation.* If we assume *additive noise*, the potential is directly related to the *stationary probability function:*

$$p_{st}(x) \propto exp\big(-U(x)\big) \qquad \text{if } \sigma(x) = \sigma \qquad \text{(S18)}$$

where $\propto$ denotes proportionality.

Probability density distribution P(x,t) of the Langevin equation

The forward Fokker-Planck equation is a partial differential equation that describes how the probability density function $P(x,t)$ of state variable *x* evolves in time.

$$\frac{\partial P(x,t)}{\partial t} = \frac{-\partial\big(D_1(x)P(x,t)\big)}{\partial x} + \frac{\partial^2\big(D_2(x)P(x,t)\big)}{\partial x^2} \qquad \text{(S19)}$$

Where D_1 is the *drift term* and D_2 the *diffusion term*. We solve this equation for the whole range of valid values of x, using reflecting boundaries. We need to define an initial probability density function, usually a narrow distribution around the initial states x_0. $P(x, t)$ approaches the stationary probability density function that is independent of the initial distribution.

In MATLAB we used the function `pdepe` to solve this partial differential equation.

Reflecting boundary conditions

A reflecting boundary is a barrier that cannot be crossed by a stochastically moving particle, instead the particle is reflected. In a partial differential equation, an analogous behavior is achieved for any state by setting a boundary condition where the spatial derivative of the dependent variable (e.g., survival $S(x,t)$ see equation S22) is constrained to zero. We use these boundary conditions for instance in the *backward Fokker Planck equation* for the borders of the domain where no shift takes place.

Stability Landscape

See: *Potential function.*

Stationary Probability Distribution

This is the probability distribution of a *Langevin equation* at stationarity ($t = \infty$). For a one-dimensional Langevin equation the stationary probability distribution, $p_{st}(x)$ is independent of the initial states (*26*, *30*) and can be calculated as:

$$p_{st}(x) \propto \frac{1}{D_2(x)} exp\left(\int^x \frac{D_1(y)}{D_2(y)} dy\right) \quad \text{(S20)}$$

Where $\propto$ denotes proportionality and y is a dummy variable and the integral has an arbitrary lower limit, D_1 is the *drift term* and D_2 the *diffusion term*. Alternatively, it can be determined by running the *forward Fokker-Planck equation* from an arbitrary initial condition until it stabilizes.

Stochastic Differential Equation

This is a differential equation that includes stochastic processes. The Langevin equation is an example of a stochastic differential equation, where the stochastic processes are separated from the deterministic processes.

Survival probability function $S(x_0, t)$

The survival probability, conditioned on initial state x_0, is the chance that system survives (meaning that it has not yet left the *basin of attraction*) up to time t given that it starts at initial state x_0:

$$S(\mathrm{t}|\mathrm{x}_0) = Pr(T > t|x_0) \quad \text{(S21)}$$

Where *Pr* is the probability that the exit time *T* exceeds time *t*, conditioned on a certain initial condition. The integral of $S(\mathrm{t}|\mathrm{x}_0)$ from time $t = 0$ to infinity is the *mean exit time* (equation S15). This function can also be used to derive the *median exit time* (equation S16) and the *distribution of exit times* (equation S9). We can determine the survival probability function by

solving the following partial differential equation for a certain *basin of attraction*, Note that we use here the notation $S(x_0, t)$, instead of $S(t|x_0)$, for convenience.

$$\frac{\partial S(x_0,t)}{\partial t} = D_1(x_0)\frac{\partial S(x_0,t)}{\partial x_0} + D_2(x_0)\frac{\partial^2 S(x_0,t)}{\partial {x_0}^2} \quad \text{(S22)}$$

$$S(x_0, 0) = 1 \quad \text{(S23)}$$

Where D_1 is the *drift term* and D_2 the *diffusion term*. In MATLAB we used the function `pdepe` to solve this partial differential equation.

Weighted mean exit time

This is the *mean exit time* $T(x_0)$ (equation S19) averaged over all initial states x_0 in a *basin of attraction* weighted with the *stationary probability density function* $p_{st}(x_0)$ (equation S18).

$$T_{av} = \int_a^b \frac{p_{st}(x_0)}{\int_a^b p_{st}(y)dy} T(x_0)dx_0 \quad \text{(S24)}$$

Where a and b are the borders of the basin of attraction. The expression $\int_a^b p_{st}(y)dy$ in the denominator is to normalize $p_{st}(x_0)$ since a weight function should integrate into 1.

Wiener process

The Wiener process is a stochastic continuous time process. The increments of a Wiener process (dW) are Gaussian distributed with a variance that is dependent on the time step. For instance, the discretized Langevin equation according to the Euler-Maruyama scheme reads:

$$x(t + \Delta t) = \Delta t f\big(x(t)\big) + \sigma\big(x(t)\big)dW(t) \quad \text{(S25)}$$

Where Δt is the time step and $dW(t)$ is a normally distributed random number with a mean of zero and standard deviation of $\sqrt{\Delta t}$.

Caption for movie S1 and S2

These videos illustrate the potential function and example trajectories of the May model (equation 4 in the main text) with parameters: a = 1.6, γ = 2.75, K = 10, r = 1.

- movie_S1.mp4 has additive noise of σ = 0.3.
- movie_S2.mp4 has additive noise of σ = 0.25.

MATLAB code and used data sets.

All analyses were done in MATLAB. This zipped archive (S1_matlabcode.zip) contains all code necessary for reproducing the plots.

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