Physics 153 Section T0H - Solution to Problem 9

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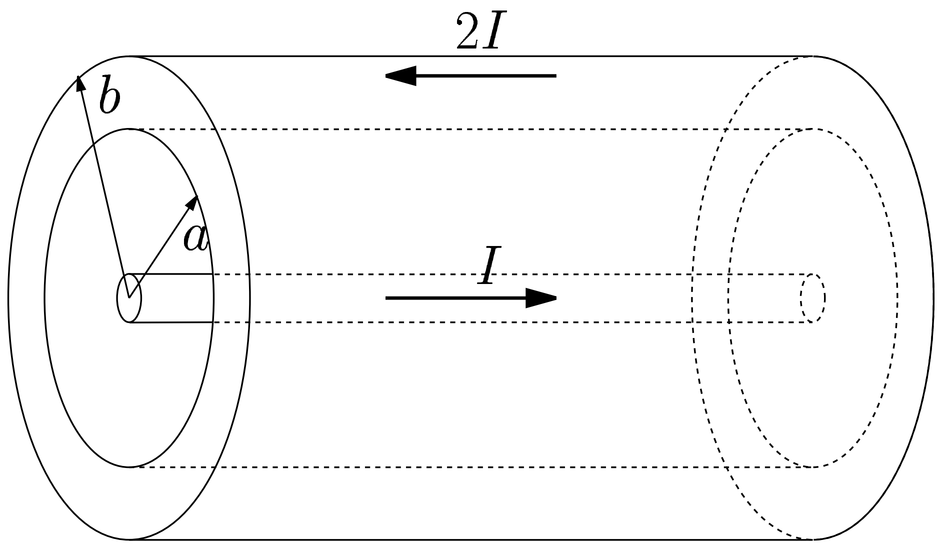
1 Assigned Problem



A very long coaxial cable consists of a thin inner wire and a concentric outer cylindrical shell with inner radius a and outer radius b. A current I runs down the inner wire and a current $2I$ comes back up the shell (uniformly distributed). Find the magnetic field $\mathbf{B}$ at a distance r from the center for (a) $r < a$, (b) $a < r < b$, and (c) $r > b$. (d) Sketch a graph of B versus r.

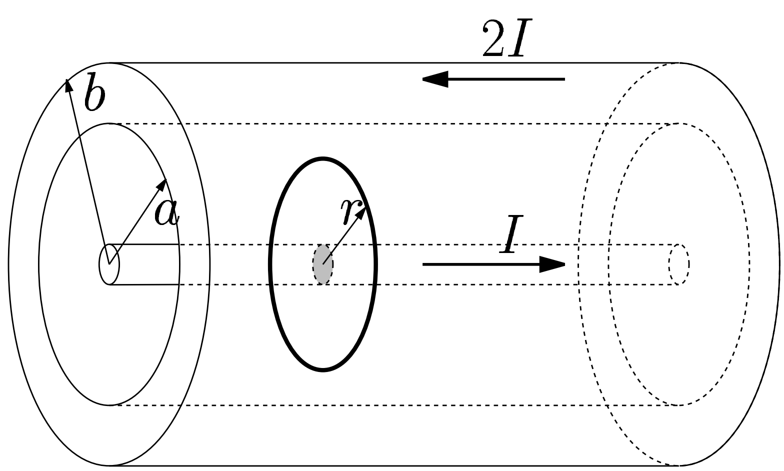
2 Solution

2.1 Part (a)

The general equation we need is Ampère's law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}. \tag{1}$$

From the right-hand rule we know the magnetic field lines wrap around the cable so the natural choice for an Ampèrian loop is a concentric circle of radius $r < a$.



This simplifies the problem a lot because, everywhere on the loop, the magnetic field has a constant strength B and is lined up with the infinitesimal line element $d\mathbf{l}$ so $\mathbf{B} \cdot d\mathbf{l} = B\,dl$ and the integral just becomes

$$\mu_0 I_{enc} = \oint \mathbf{B} \cdot d\mathbf{l} \tag{2}$$

$$= B \oint dl \tag{3}$$

$$= B(2\pi r) \tag{4}$$

so the magnetic field is just

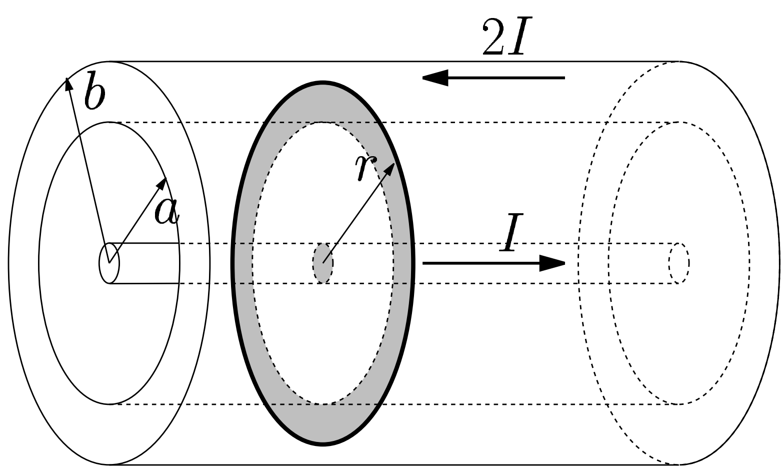
$$B(r < a) = \frac{\mu_0 I}{2\pi r}. \tag{5}$$

The direction, given by the right-hand rule, is clockwise and I'll take that to be positive for the remainder of the problem.

Note that I_{enc} in Ampère's law is just the current passing through the loop which is I in this case.

2.2 Part (b)

This part gets a little more complicated. The basic procedure is the same—draw an Ampèrian loop for $a < r < b$ and apply Ampère's law—except calculating the current inside the loop is a bit trickier.



We know we have a contribution I in one direction from the inner wire but we also have some of the current from the outer shell (however much is inside the loop). Recall that the current is uniformly-distributed in the shell so we can calculate the cross-sectional current density J, which is just the current per unit area (ie. the total current divided by the total area):

$$J = \frac{-2I}{\pi(b^2 - a^2)}. \tag{6}$$

Multiplying the current density by the area of the shell which is inside the loop gives the total current inside the loop (plus the inner wire):

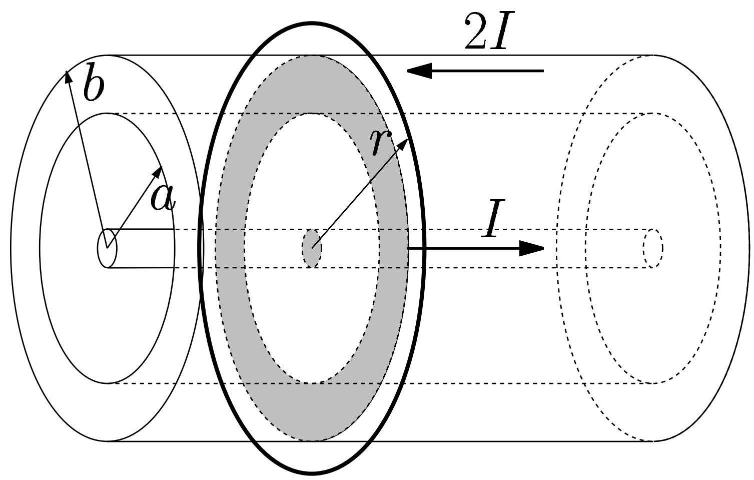
$$\begin{aligned} I_{enc} &= I + J\pi(r^2 - a^2) & (7) \\ &= I - 2I\frac{r^2 - a^2}{b^2 - a^2}. & (8) \end{aligned}$$

The solution to Ampère's law is identical to Eq. 4 so

$$\begin{aligned} B(a < r < b) &= \frac{\mu_0}{2\pi r} I_{enc} & (9) \\ &= \frac{\mu_0}{2\pi r}\left[I - 2I\frac{r^2 - a^2}{b^2 - a^2}\right]. & (10) \end{aligned}$$

2.3 Part (c)

Ok, now it gets easier again because the Ampèrian loop is outside of the shell so the enclosed current is constant.



Again, using Eq. 4 but this time with $I_{enc} = I - 2I = -I$ gives

$$B(r > b) = -\frac{\mu_0 I}{2\pi r}. \tag{11}$$

The minus sign means the magnetic field is in the opposite direction to that found in Part (a), counter-clockwise.

2.4 Part (d)

The net field (measured as positive clockwise) just comes from Eqs. 5, 10, and 11 in each regime. The first and last are pretty easy to sketch because they are both $1/r$-type functions and, roughly speaking, the dependence in the middle area is close to linear (and it crosses zero somewhere). So the final sketch should look something like this: