Physics 153 Section T0H - Solution to Problem 8

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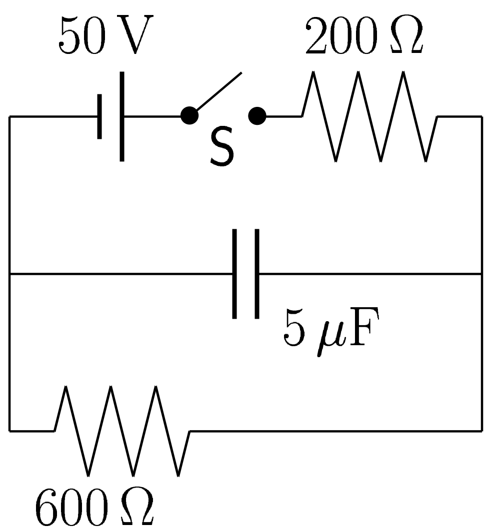
1 Assigned Problem

(From Tipler Ch. 23 #60.)

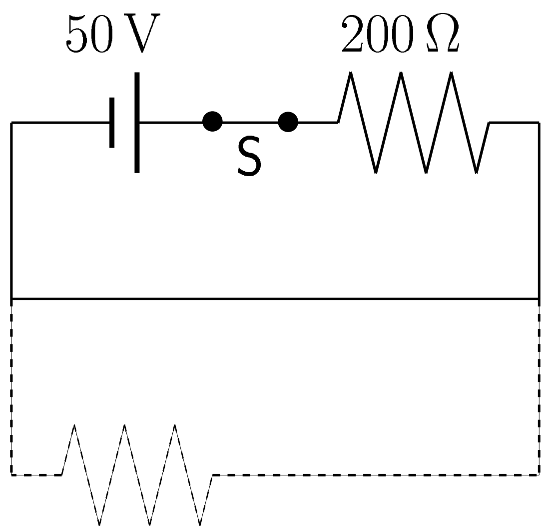


For the above circuit, (a) what is the initial battery current immediately after switch S is closed? (b) What is the battery current a long time after switch S is closed? (c) What is the current in the $600\,\Omega$ resistor as a function of time?

2 Solution

2.1 Part (a)

Initially after the switch is closed, there is no charge on the capacitor so it acts like a short circuit and none of the current goes through the $600\,\Omega$ resistor.

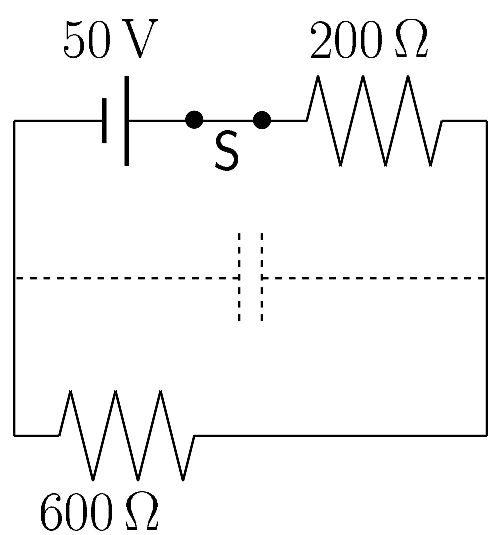


So the circuit just consists of a battery and a resistor, giving a current across the battery at time 0

$$I_V(0) = \frac{V}{R} = \frac{50}{200} = 250\,\text{mA}. \tag{1}$$

2.2 Part (b)

After a long time the charge on the capacitor has built up and the current goes to zero. So, effectively, the circuit looks like this:



So we just have two resistors in series with a battery giving a current across the battery at time ∞

$$I_V(\infty) = \frac{V}{R} = \frac{50}{800} = 62.5\,\mathrm{mA}. \tag{2}$$

2.3 Part (c)

Now we want the current across the resistor (not the battery). First, recall the general equation for a RC circuit is

$$I(t) - I(\infty) = [I(0) - I(\infty)]e^{-t/\tau}. \tag{3}$$

Across the $600\,\Omega$ resistor the initial and final currents are

$$I_{600}(0) = 0\,\mathrm{A}, \tag{4}$$
$$I_{600}(\infty) = I_V(\infty) = 62.5\,\mathrm{mA}. \tag{5}$$

2.3.1 Time constant

Calculating the time constant turns out to be *a lot* trickier than I might have told you in class. I thought you could just neglect the $600\,\Omega$ resistor because initially all the current just went through the other one.

But it turns out you can't. Actually, some of you had the right intuition: you have to treat the resistors as if they are in parallel and calculate the equivalent resistance

$$R_{eq} = \left[\frac{1}{600} + \frac{1}{200}\right]^{-1} \tag{6}$$

$$= 150\,\Omega \tag{7}$$

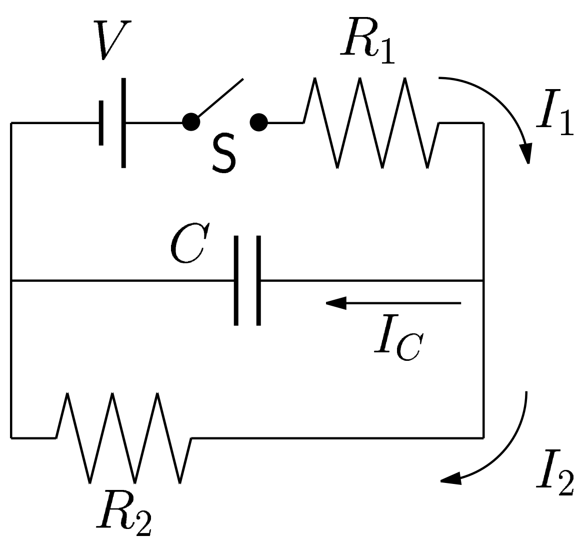
to give a time constant of

$$\tau = R_{eq}C = 150 \cdot 5 \times 10^{-6} = 750\,\mu\text{s}. \tag{8}$$

To see why this works you have to start from first principles and solve the differential equation. I'll do that here for the curious but you won't be expected to know this. If you don't care, just skip on down to the final solution.

2.3.2 Differential equation

Ok, let's label everything so we don't have to substitute in the numbers.



I_C is just the rate charge passes through (or builds up on) the capacitor so

$$I_C = \frac{dQ}{dt} \tag{9}$$

and for any capacitor the voltage across it is

$$V_C = Q/C. \tag{10}$$

The other equations we need to know come from Kirchhoff's laws:

$$I_1 = I_C + I_2, \tag{11}$$

$$V - I_1 R_1 - V_C = 0, \tag{12}$$

$$V_C - I_2 R_2 = 0. \tag{13}$$

Plugging in I_C and V_C gives (after a little work)

$$\frac{dQ}{dt} = \frac{V}{R_1} - \frac{Q}{R_{eq} C} \tag{14}$$

where R_{eq} is the equivalent resistance of R_1 and R_2 in parallel.

I don't know how to solve that directly but if I make a change of variables

$$y = Q - \frac{V}{R_1} R_{eq} C \tag{15}$$

then the equation reduces to

$$\frac{dy}{dt} = -\frac{1}{R_{eq}C}\, y \tag{16}$$

$$\frac{dy}{y} = -\frac{dt}{R_{eq}C} \tag{17}$$

$$\ln y = -\frac{t}{R_{eq}C} + K_1 \tag{18}$$

(where K_1 is an integration constant) which has the solution

$$y = K_2 e^{-\frac{t}{R_{eq}C}} \tag{19}$$

where the integration constant has been moved into K_2.

We already can see that the time constant for this circuit is

$$\tau = R_{eq}C \tag{20}$$

but let's continue to solve for $I_2(t)$.

Changing variables back to Q gives

$$Q = K_2 e^{-t/\tau} + \frac{V}{R_1} R_{eq} C. \tag{21}$$

Solving for the constant K_2 is pretty easy if we remember that $Q(0) = 0$ so

$$K_2 = -\frac{V}{R_1} R_{eq} C \tag{22}$$

so

$$Q(t) = \frac{V}{R_1} R_{eq} C \left(1 - e^{-t/\tau}\right). \tag{23}$$

Now, from Eq. (13) we get

$$\begin{aligned} I_2(t) &= \frac{Q}{R_2 C} && (24) \\ &= \frac{V R_{eq}}{R_1 R_2} \left(1 - e^{-t/\tau}\right) && (25) \\ &= \frac{V}{R_1 + R_2} \left(1 - e^{-t/\tau}\right). && (26) \end{aligned}$$

Phew! All that just to justify that you have to use the resistors in parallel to calculate the time constant!

2.3.3 Final solution

So the current through the resistor as a function of time is

$$I_{600}(t) = (62.5\,\mathrm{mA})\left(1 - e^{-t/(750\,\mu\mathrm{s})}\right). \tag{27}$$