Asymptotic approximation of exponential integrals

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Contents

1 Preliminaries **1**
1.1 What this is . . . 1
1.2 Problem statement . . . 1
1.3 Big-O and small-o notation . . . 1
1.4 Definition: Asymptotic approximation . . . 2
1.5 Definition: Globally isolated minimum . . . 2

2 Statements **2**
2.1 Exponents linear in T, increasing in z . . . 2
2.2 Exponents linear in T, strongly-concave in z (Laplace's method on $\mathbb{R}$) . . . 3
2.3 Exponents linear in T (Laplace's method on finite intervals) . . . 5
2.4 Exponents linear in T (Laplace's method on $\mathbb{R}$) . . . 7

1 Preliminaries

1.1 What this is

This article is a personal collection of some theorems and rigorous proofs for leading order asymptotic approximations of so called *exponential integrals*, basically special cases of Laplace's method. I do not claim any originality of the theorems themselves. If you find any errors, please let me know. A good introduction into the subject can be found in [1, chapter 3].

1.2 Problem statement

Given two functions $F = F(T, z)$ and $g = g(z)$, as well as $-\infty \leq a \leq b \leq \infty$, find asymptotic expansions for the integral

$$J(T) := \int_a^b e^{F(T,z)} g(z)\ dz. \tag{1.1}$$

We assume at all times the integrant to be in $L_1(a, b)$.

1.3 Big-O and small-o notation

For any two functions $f, g : \mathbb{R}_+ \to \mathbb{C}$ we shall write $f(T) \in \mathcal{O}(g(T))$ as $T \to \infty$, whenever there exists a constant $C \geq 0$ such that $|f(T)| \leq C \cdot |g(T)|$ for T large enough. We shall write $f(T) \in o(g(T))$ as $T \to \infty$, if for every $\varepsilon > 0$ one has $|f(T)| \leq \varepsilon \cdot |g(T)|$ provided that T is large enough (the threshold depending on ε).

Note: If $g(T) \neq 0$ for all $T \geq 0$, then $f(T) \in o(g(T))$ as $T \to \infty$ is equivalent to saying $f(T)/g(T) \overset{T\to\infty}{\longrightarrow} 0$.

1.4 Definition: Asymptotic approximation

Let $J, J_o : (0,\infty) \to \mathbb{C}$ be functions such that $(J(T) - J_o(T)) \in o(J_o(T))$ as $T \to \infty$. Then J_o is called a **leading order asymptotic approximation** of $J(T)$ as $T \to \infty$. We write $J(T) \sim J_o(T)$ as $T \to \infty$.

Let $J_0, .., J_n : (0,\infty) \to \mathbb{C}$ be functions such that $J_n(T) \in o(J_{n-1}(T))$, $\forall n \in \{1, .., n\}$ and such that $J(T) - \sum_{k=0}^n J_k(T) \in o(J_n(T))$. Then the sum $\sum_{k=0}^n J_k(T)$ is called an **n-th order asymptotic approximation** of $J(T)$. We write $J(T) \sim \sum_{k=0}^n J_k(T)$.

Let $J_0, J_1, .. : (0,\infty) \to \mathbb{C}$ be functions such that $J(T) - \sum_{k=0}^n J_k(T) \in o(J_n(T))$ for all $n \in \mathbb{N}_0$. Then the formal series $\sum_{k=0}^\infty J_k(T)$ is called an **asymptotic expansion** of $J(T)$. We write $J(T) \sim \sum_{k=0}^\infty J_k(T)$. Note that in that case also $J_{n+1}(T) \in o(J_n(T))$ for all $n \in \mathbb{N}_0$.

Rules: Leading order asymptotic approximation is an equivalence relation among functions.

1. **Reflexivity:** $J(T) \sim J(T)$.
2. **Symmetry:** If $J(T) \sim J_o(T)$ then also $J_o(T) \sim J(T)$.
3. **Transitivity:** If $J(T) \sim J_o(T)$ and $J_o(T) \sim J_1(T)$, then also $J(T) \sim J_1(T)$.
4. **Product rule:** If $f(T) \sim \tilde{f}(T)$ and $g(T) \sim \tilde{g}(T)$, then also $(fg)(T) \sim (\tilde{f}\tilde{g})(T)$.

1.5 Definition: Globally isolated minimum

Let X be a metric space and $f : X \to \mathbb{R}$ some function. We shall say that $x_o \in X$ is a **globally isolated minimum** of f if it is the only point in X satisfying $f(x_o) = \inf_{x\in X} f(x)$ and for any neighborhood U of x_o one has $f(x) < \inf_{x\in X\setminus U} f(x)$.

2 Statements

2.1 Exponents linear in T, increasing in z

Consider the special case $(a,b) = (0,\infty)$, $F(T,z) = -T \cdot f(z)$ for some $f : [0,\infty) \to \mathbb{R}$. Assume f to be differentiable with $f'(z) > 0$ for all $z \geq 0$. Define $\zeta(u) := f^{-1}(u + f(0))$ and $\Omega(u) := g(\zeta(u))/f'(\zeta(u))$ for $u \geq 0$. Assume the function $\Omega : [0,\infty) \to \mathbb{R}$ to be continuous at the origin and N times differentiable at the origin ($N \in \mathbb{N}_0$), with $\Omega^{(N)}(0) \neq 0$. Assume that Ω increases at most exponentially with u, that is $|\Omega(u)| \leq Ce^{qu}$ for some $0 \leq q < \infty$ and $C \geq 0$. Then

$$J(T) := \int_0^\infty e^{-Tf(z)} g(z)\, dz \sim e^{-Tf(0)} \cdot \sum_{\substack{0\leq n\leq N \\ \Omega^{(n)}(0)\neq 0}} \frac{\Omega^{(n)}(0)}{T^{n+1}}. \tag{2.1}$$

Proof: Define $u(z) := f(z) - f(0)$, $v := Tu$ and denote $\beta := f(\infty) - f(0) \geq 0$, then

$$J(T) = e^{-Tf(0)} \int_0^\beta e^{-Tu} \cdot \Omega(u)\, du = \frac{e^{-Tf(0)}}{T} \int_0^{T\beta} e^{-v} \cdot \Omega\left(\tfrac{v}{T}\right)\, dv. \tag{2.2}$$

Note that by assumption, there exists a function $R_N : \mathbb{R}_+ \to \mathbb{R}$ such that $R(u) \to 0$ as $u \to \infty$ and

$$\Omega(u) = \sum_{n=0}^N \frac{\Omega^{(n)}(0)}{n!} \cdot u^n + u^N R_N(u). \tag{2.3}$$

Using (2.3) in (2.2) we can write

$$\begin{aligned} J(T) =& \frac{e^{-Tf(0)}}{T} \int_0^{T\beta} e^{-v} \cdot \left[\sum_{n=0}^N \frac{\Omega^{(n)}(0)}{n! \cdot T^n} \cdot v^n + \frac{v^N}{T^N} R_N\left(\tfrac{v}{T}\right)\right]\, dv \\ =& \frac{e^{-Tf(0)}}{T} \left[\sum_{n=0}^N \frac{\Omega^{(n)}(0)}{n! \cdot T^n} \cdot \int_0^{T\beta} e^{-v} \cdot v^n\, dv + \frac{1}{T^N} \int_0^{T\beta} v^N e^{-v} \cdot R_N\left(\tfrac{v}{T}\right)\right]. \end{aligned} \tag{2.4}$$

Note that

$$\int_0^{T\beta} e^{-v} \cdot v^n \, dv \sim \int_0^{\infty} e^{-v} \cdot v^n \, dv = n! \tag{2.5}$$

as $T \to \infty$. Left to show is that $I(T) := \int_0^{T\beta} v^N e^{-v} \cdot R_N(v/T) \, dv \to 0$ as $T \to \infty$. We split

$$I(T) = \underbrace{\int_0^{\sqrt{T}} v^N e^{-v} \cdot R_N(v/T) \, dv}_{=:A(T)} + \underbrace{\int_{\sqrt{T}}^{T\beta} v^N e^{-v} \cdot R_N(v/T) \, dv}_{=:B(T)}. \tag{2.6}$$

We observe that

$$|A(T)| \leq \sup_{u \in [0,1/\sqrt{T}]} |R_N(u)| \cdot \int_0^{\infty} v^N e^{-v} \, dv \overset{T\to\infty}{\longrightarrow} 0. \tag{2.7}$$

Note that by assumption on the growth rate of Ω, we know $R_N(u) \leq Ce^{qu}$ for suitable constants $C \geq 0$, $0 \leq q < 1$. We can therefore estimate

$$|B(T)| \leq C \int_{\sqrt{T}}^{\infty} v^N e^{-v(1-q/T)} \, dv \overset{T\to\infty}{\longrightarrow} 0. \tag{2.8}$$

Note that we used the fact that $1 - q/T > 1/2$ for T sufficiently large. This completes the proof. □

Example: Let $g : \mathbb{R}_+ \to \mathbb{R}$ be both continuous and N times differentiable at the origin ($N \in \mathbb{N}_0$), with $g^{(N)}(0) \neq 0$. Assume g increases at most exponentially with u. Then

$$\int_0^{\infty} e^{-Tz} g(z) \, dz \sim \sum_{\substack{0 \leq n \leq N \\ f^{(n)}(0) \neq 0}} \frac{g^{(n)}(0)}{T^{n+1}}. \tag{2.9}$$

2.2 Exponents linear in T, strongly-concave in z (Laplace's method on $\mathbb{R}$)

Consider the special case $(a, b) = (-\infty, \infty)$, $F(T, z) = -T \cdot f(z)$ for some $f : \mathbb{R} \to \mathbb{R}$. Assume f to have a global minimum at z_o, where it is twice differentiable with $f'(z_o) = 0$ and $f''(z_o) > 0$. Assume

$$f(z) = f(z_o) + \frac{f''(z_o)}{2} \cdot (z - z_o)^2 + r_2(z - z_o), \tag{2.10}$$

where $r_2(x) \in \mathcal{O}(x^3)$ as $x \to 0$. Assume

$$r_2(x) \geq -\frac{q}{2} f''(z_o) \cdot x^2 \quad \forall x \in \mathbb{R} \tag{2.11}$$

for some $0 \leq q < \infty$. Assume $g(z)$ to be continuous in z_o with $g(z_o) \neq 0$, such that

$$|g(z_o + x) - g(z_o)| \leq C \exp\left[T\left(qx^2 f''(z_o)/2 + r_2(x)\right)\right] \quad \forall x \in \mathbb{R} \tag{2.12}$$

for some $0 \leq q < 1$, $C \geq 0$ and T large enough. Then

$$J(T) := \int_{-\infty}^{\infty} e^{-Tf(z)} g(z) \, dz \sim \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \int_{-\infty}^{\infty} e^{-u^2} \cdot g(z_o) \, dz \tag{2.13}$$

as $T \to \infty$.

Proof: Using (2.10) we can write

$$J(T) = e^{-Tf(z_o)} \cdot \int_{-\infty}^{\infty} e^{-\frac{T}{2}f''(z_o)(z-z_o)^2 - Tr_2(z-z_o)} g(z)\ dz. \tag{2.14}$$

Set the integration variable to $u := \sqrt{f''(z_o)/2} \cdot (z - z_o)$ and define

$$G(u) := g(z_o + u\sqrt{2/f''(z_o)}), \quad R_2(u) := r_2(u\sqrt{2/f''(z_o)}), \tag{2.15}$$

so that

$$\begin{aligned} J(T) =& \frac{e^{-Tf(z_o)}}{\sqrt{f''(z_o)/2}} \cdot \int_{-\infty}^{\infty} e^{-Tu^2 - TR_2(u)} \cdot G(u)\ du \\ =& \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \int_{-\infty}^{\infty} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot G\big(v/\sqrt{T}\big)\ dv. \end{aligned} \tag{2.16}$$

As g is continuous in z_o we can write $G(u) = g(z_o) + Q_0(u)$ for some $Q_0(u) \in o(u^0)$ as $u \to 0$. Hence

$$\begin{aligned} J(T) = \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \Bigg[\int_{-\infty}^{\infty} e^{-v^2} \cdot g(z_o)\ dv + \int_{-\infty}^{\infty} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot Q_0(v/\sqrt{T})\ dv \\ + \int_{-\infty}^{\infty} e^{-v^2} \cdot g(z_o) \cdot \Big[e^{-TR_2(v/\sqrt{T})} - 1 \Big]\ dv \Bigg]. \end{aligned} \tag{2.17}$$

Left to show is that

$$I(T) := \underbrace{\int_{-\infty}^{\infty} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot Q_0(v/\sqrt{T})\ dv}_{=:A(T)} + \underbrace{\int_{-\infty}^{\infty} e^{-v^2} \cdot g(z_o) \cdot \Big[e^{-TR_2(v/\sqrt{T})} - 1 \Big]\ dv}_{=:B(T)} \tag{2.18}$$

vanishes as $T \to 0$. Define $\gamma(T) := T^{\frac{1}{7}}$, then

$$\begin{aligned} |A(T)| \leq& \int_{B_{\gamma(T)}(0)} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot \Big| Q_0(v/\sqrt{T}) \Big|\ dv + \int_{\mathbb{R}\setminus B_{\gamma(T)}(0)} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot \Big| Q_0(v/\sqrt{T}) \Big|\ dv \\ \leq& \underbrace{\sup_{u \in B_{T^{-5/14}}(0)} \Big[e^{-TR_2(u)} \cdot |Q_0(u)| \Big]}_{\overset{T\to\infty}{\longrightarrow} 0} \cdot \int_{\mathbb{R}} e^{-v^2}\ dv + \underbrace{\int_{\mathbb{R}\setminus B_{\gamma(T)}(0)} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot \Big| Q_0(v/\sqrt{T}) \Big|\ dv}_{\tilde{A}(T)}. \end{aligned} \tag{2.19}$$

From (2.11) and (2.28) we can (for T large enough) estimate

$$\begin{aligned} \tilde{A}(T) \leq& \int_{\mathbb{R}\setminus B_{\gamma(T)}(0)} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot C e^{qv^2 + TR_2\left(v/\sqrt{T}\right)}\ dv \\ =& C \int_{\mathbb{R}\setminus B_{\gamma(T)}(0)} e^{-(1-q)v^2}\ dv \overset{T\to\infty}{\longrightarrow} 0. \end{aligned} \tag{2.20}$$

Similarly

$$|B(T)| \leq \underbrace{\sup_{u \in B_{T^{-5/14}}(0)} \Big| e^{-TR_2(u)} - 1 \Big|}_{\overset{T\to\infty}{\longrightarrow} 0} \cdot \int_{\mathbb{R}} e^{-v^2} \cdot |g(z_o)|\ dv + \underbrace{\int_{B_{\gamma(T)}(0)} e^{-v^2} \cdot |g(z_o)| \cdot \Big| e^{-TR_2(v/\sqrt{T})} - 1 \Big|}_{=:\tilde{B}(T)}. \tag{2.21}$$

Note that from condition (2.11) we can estimate $e^{-TR_2(v/\sqrt{T})} \leq e^{qv^2}$, so that for T large enough

$$\tilde{B}(T) \leq 2\,|g(z_o)| \int_{B_{\gamma(T)}(0)} e^{-(1-q)v^2}\ dv \overset{T\to\infty}{\longrightarrow} 0. \tag{2.22}$$

This finishes the proof. □

2.3 Exponents linear in T (Laplace's method on finite intervals)

Consider the special case $-\infty < a \le b < \infty$, $F(T,z) = -T \cdot f(z)$ for some $f : [a,b] \to \mathbb{R}$. Assume f to to have a globally isolated minimum at $z_o \in (a,b)$ (see def. 1.5), where it is three times differentiable with $f'(z_o) = 0$ and $f''(z_o) > 0$. Assume g to be L_2-integrable over $[a,b]$, continuous in z_o with $g(z_o) \neq 0$ and such that $(g(z_o + x) - g(z_o)) \in \mathcal{O}(x^\lambda)$ as $x \to 0$ for some $\lambda > 0$. Then

$$J(T) := \int_a^b e^{-Tf(z)} g(z)\, dz \sim \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \int_{-\infty}^{\infty} e^{-u^2} \cdot g(z_o)\, dz \tag{2.23}$$

as $T \to \infty$.

Proof: Let $r_2 : \mathbb{R} \to \mathbb{R}$ be such that

$$f(z) = f(z_o) + \frac{f''(z_o)}{2} \cdot (z - z_o)^2 + r_2(z - z_o), \tag{2.24}$$

whereas $r_2(x) \in \mathcal{O}(x^3)$ as $x \to 0$.

Claim 01: There exists $0 \le q < 1$ such that $r_2(x) \ge -q\frac{x^2}{2} f''(z_o)$ for all $x \in \Omega := [a - z_o, b - z_o]$.

> **Proof of claim:** Choose $\varepsilon > 0$ so small that $|r_2(x)| \le \frac{x^2}{4} f''(z_o)$ for all $|x| \le \varepsilon$. Let $A := \inf_{x \in \Omega \setminus B_\varepsilon(0)} [f(z_o + x) - f(z_o)]$. As f is has a globally isolated minimum in z_o, we know $A > 0$. Choose $\frac{1}{2} \le q < 1$ so large that $(1-q)\frac{x^2}{2} f''(z_o) \le A$ for all $x \in \Omega$.
>
> Consequently, for all $x \in \Omega \setminus B_\varepsilon(0)$ one has
>
> $$r_2(x) = f(z_o + x) - f(z_o) - \frac{x^2}{2} f''(z_o) \ge (1-q)\frac{x^2}{2} f''(z_o) - \frac{x^2}{2} f''(z_o) = -q\frac{x^2}{2} f''(z_o). \tag{2.25}$$
>
> For $x \in B_\varepsilon(0) \cap \Omega$ one has
>
> $$r_2(x) \ge -\frac{1}{2}\frac{x^2}{2} f''(z_o) \ge -q\frac{x^2}{2} f''(z_o). \tag{2.26}$$
>
> This proves the claim.

Using (2.24) we can write

$$J(T) = e^{-Tf(z_o)} \cdot \int_a^b e^{-\frac{T}{2} f''(z_o)(z - z_o)^2 - Tr_2(z - z_o)} g(z)\, dz. \tag{2.27}$$

Set the integration variable to $u := \sqrt{f''(z_o)/2} \cdot (z - z_o)$ and define

$$\begin{aligned} &G(u) := g(z_o + u\sqrt{2/f''(z_o)}), \quad R_2(u) := r_2(u\sqrt{2/f''(z_o)}), \\ &\alpha := \sqrt{f''(z_o)/2} \cdot (a - z_o), \quad \beta := \sqrt{f''(z_o)/2} \cdot (b - z_o) \end{aligned} \tag{2.28}$$

so that

$$\begin{aligned} J(T) &= \frac{e^{-Tf(z_o)}}{\sqrt{f''(z_o)/2}} \cdot \int_\alpha^\beta e^{-Tu^2 - TR_2(u)} \cdot G(u)\, du \\ &= \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \int_{\alpha\sqrt{T}}^{\beta\sqrt{T}} e^{-v^2} e^{-TR_2(v/\sqrt{T})} \cdot G(v/\sqrt{T})\, dv. \end{aligned} \tag{2.29}$$

By assumption on g we can write $G(u) = g(z_o) + Q_0(u)$ for some $Q_0(u) \in \mathcal{O}(u^\lambda)$ as $u \to 0$ (with $\lambda > 0$). Hence

$$\begin{aligned} J(T) = \frac{e^{-Tf(z_o)}}{\sqrt{Tf''(z_o)/2}} \cdot \Bigg[&\int_{\alpha\sqrt{T}}^{\beta\sqrt{T}} e^{-v^2} \cdot g(z_o)\, dv + \sqrt{T} \int_\alpha^\beta e^{-Tu^2} e^{-TR_2(u)} \cdot Q_0(u)\, du \\ &+ \sqrt{T} \int_\alpha^\beta e^{-Tu^2} \cdot g(z_o) \cdot \left[e^{-TR_2(u)} - 1 \right] du \Bigg]. \end{aligned} \tag{2.30}$$

Note that

$$\int_{\alpha\sqrt{T}}^{\beta\sqrt{T}} e^{-v^2}\cdot g(z_o)\ dv \sim \int_{\infty}^{\infty} e^{-v^2}\cdot g(z_o)\ dv \tag{2.31}$$

as $T\to\infty$. Left to show is that

$$I(T) := \underbrace{\sqrt{T}\int_{\alpha}^{\beta} e^{-Tu^2}e^{-TR_2(u)}\cdot Q_0(u)\ du}_{=:A(T)} + \underbrace{\sqrt{T}\int_{\alpha}^{\beta} e^{-Tu^2}\cdot g(z_o)\cdot\left[e^{-TR_2(u)}-1\right]\ du}_{=:B(T)} \tag{2.32}$$

vanishes as $T\to 0$. Note that by claim 01 we know $R_2(u)\geq -qu^2$ and thus

$$|A(T)| \leq \sqrt{T}\int_{\alpha}^{\beta} e^{-T(1-q)u^2}\cdot|Q_0(u)|\ du. \tag{2.33}$$

Choose any $\frac{1}{2(\lambda+1)} < \mu < \frac{1}{2}$ and set $\gamma(T) := T^{-\mu}$. Then using (2.33) we estimate

$$\begin{aligned}|A(T)| \leq& \sqrt{T}\int_{B_{\gamma(T)}(0)} \underbrace{e^{-T(1-q)u^2}}_{\leq 1}\cdot|Q_0(u)|\ du + \sqrt{T}\int_{(\alpha,\beta)\setminus B_{\gamma(T)}(0)} e^{-T(1-q)u^2}\cdot|Q_0(u)|\ du\\ \leq& \underbrace{\sqrt{T}\cdot 2\gamma(T)\sup_{u\in B_{\gamma(T)}(0)}|Q_0(u)|}_{\in\mathcal{O}(T^{\frac{1}{2}-\mu(1+\lambda)})\subseteq o(T^0)\overset{T\to\infty}{\longrightarrow}0} + \|Q_0\|_{L_2}\cdot\underbrace{\sqrt{T}\int_{(\alpha,\beta)\setminus B_{\gamma(T)}(0)} e^{-2T(1-q)u^2}\ du}_{=:\tilde{A}(T)}.\end{aligned} \tag{2.34}$$

Note that

$$\begin{aligned}\tilde{A}(T) =& \frac{1}{\sqrt{2(1-q)}}\int_{\left(\sqrt{2(1-q)T}\cdot(\alpha,\beta)\right)\setminus B_{T^{\frac{1}{2}-\mu}\sqrt{2(1-q)}}(0)} e^{-y^2}\ dy\\ =& \frac{\pi}{\sqrt{2(1-q)}}\cdot\Big[\underbrace{\operatorname{Erf}\left(\sqrt{2T(1-q)}\,|\alpha|\right)}_{\overset{T\to\infty}{\longrightarrow}1} - \underbrace{\operatorname{Erf}\left(\sqrt{2(1-q)}\cdot T^{\frac{1}{2}-\mu}\right)}_{\overset{T\to\infty}{\longrightarrow}1} + \underbrace{\operatorname{Erf}\left(\sqrt{2T(1-q)}\,|\beta|\right)}_{\overset{T\to\infty}{\longrightarrow}1} - \underbrace{\operatorname{Erf}\left(\sqrt{2(1-q)}\cdot T^{\frac{1}{2}-\mu}\right)}_{\overset{T\to\infty}{\longrightarrow}1}\Big],\end{aligned} \tag{2.35}$$

so that $\tilde{A}(T)\overset{T\to\infty}{\longrightarrow}0$. In (2.35) we used the fact that $T^{\frac{1}{2}-\mu}\overset{T\to\infty}{\longrightarrow}\infty$ since $\mu<\frac{1}{2}$. Similarly, we let $\delta(T) := T^{-7/16}$ and split

$$\begin{aligned}|B|(T) \leq& |g(z_o)|\sqrt{T}\int_{\alpha}^{\beta} e^{-Tu^2}\left|e^{-TR_2(u)}-1\right|\ du = |g(z_o)|\int_{\alpha\sqrt{T}}^{\beta\sqrt{T}} e^{-v^2}\cdot\left|e^{-TR_2(v/\sqrt{T})}-1\right|\ dv\\ \leq& |g(z_o)|\int_{B_{\gamma(T)}(0)}\underbrace{e^{-v^2}}_{\leq 1}\left|e^{-TR_2(v/\sqrt{T})}-1\right|\ dv + |g(z_o)|\int_{(\sqrt{T}\alpha,\sqrt{T}\beta)\setminus B_{T^{1/16}}(0)} e^{-v^2}\cdot\left|e^{-TR_2(v/\sqrt{T})}-1\right|\ dv\\ \leq& \underbrace{|g(z_o)|\sqrt{T}2\gamma(T)\sup_{u\in B_{\gamma(T)}(0)}\left|e^{-TR_2(u)}-1\right|}_{\in\mathcal{O}(T^{-1/4})\overset{T\to\infty}{\longrightarrow}0} + \underbrace{|g(z_o)|\int_{(\sqrt{T}\alpha,\sqrt{T}\beta)\setminus B_{T^{1/16}}(0)} e^{-v^2}\cdot\left|e^{-TR_2(v/\sqrt{T})}-1\right|\ dv}_{=:\tilde{B}(T)}\end{aligned} \tag{2.36}$$

Using $R_2(u)\geq -qu^2$ we can estimate

$$\tilde{B}(T) \leq 2\int_{\mathbb{R}\setminus B_{T^{1/16}}(0)} e^{-(1-q)v^2}\ dv \overset{T\to\infty}{\longrightarrow} 0. \tag{2.37}$$

This finishes the proof. □

2.4 Exponents linear in T (Laplace's method on $\mathbb{R}$)

Consider the special case $(a,b) = (-\infty,\infty)$, $F(T,z) = -T\cdot f(z)$ for some $f:\mathbb{R}\to\mathbb{R}$. Assume f to have a globally isolated minimum at $z_o\in\mathbb{R}$, where it is three times differentiable with $f'(z_o)=0$ and $f''(z_o)>0$. Assume $g(z)$ to be L_2-integrable on any compact interval, continuous in z_o with $g(z_o)\neq 0$ and such that $(g(z_o+x)-g(z_o))\in\mathcal{O}(x^\lambda)$ as $x\to 0$ for some $\lambda>0$. . Assume that at least one of the following hold:

1. g is L_1-integrable on $\mathbb{R}$.
2. $f(z)$ eventually grows at least linearly with z (i.e. $\exists C>0$ such that $f(z)>C\,|z|$ for z large enough) and $g(z)$ grows at most exponentially (i.e. $\exists 0\leq q<\infty$ such that $g(z)\in\mathcal{O}(e^{q|z|})$).

Then

$$J(T) := \int_{-\infty}^{\infty} e^{-Tf(z)}g(z)\;dz \sim \frac{e^{-Tf(z_o)}g(z_o)}{\sqrt{Tf''(z_o)/2}}\cdot\int_{-\infty}^{\infty} e^{-u^2}\;dz \tag{2.38}$$

as $T\to\infty$.

Proof: All cases are generalizations and immediate consequences of theorem 2.3.

1. Choose any $-\infty<a<z_o<b<\infty$. By 2.3 it suffices to show that

$$\int_{\mathbb{R}\setminus[a,b]} e^{-Tf(z)}g(z)\;dz \in o\left(\frac{e^{-Tf(z_o)}}{\sqrt{T}}\right) \tag{2.39}$$

as $T\to\infty$. Let $\lambda := \inf_{x\in\mathbb{R}\setminus(a,b)} f(x)$. Then since z_o is a globally isolated minimum of f, we know $\lambda > f_o$. We can therefore estimate

$$\left|\int_{\mathbb{R}\setminus[a,b]} e^{-Tf(z)}g(z)\;dz\right| \leq \int_{\mathbb{R}\setminus[a,b]} e^{-Tf(z)}\,|g(z)|\;dz \leq e^{-T\lambda}\,\|g\|_{L_1}\in o\left(\frac{e^{-Tf(z_o)}}{\sqrt{T}}\right). \tag{2.40}$$

2. Choose $-\infty<a<z_o<b<\infty$ such that $f(z_o+x)-f(z_o)\geq\mu\cdot|x|$ and $|g(z_o+x)|\leq G_o\cdot e^{T\lambda|x|}$ for constants $0<\lambda<\mu$, $0<G_o$ and all $(z_o+x)\notin[a,b]$, provided T is large enough. By 2.3 it suffices to show that

$$\int_{\mathbb{R}\setminus[a,b]} e^{-T(f(z)-f(z_o))}g(z)\;dz\in o\big(T^{-\frac{1}{2}}\big) \tag{2.41}$$

as $T\to\infty$. We can indeed estimate

$$\begin{aligned}\left|\int_{\mathbb{R}\setminus[a,b]} e^{-T(f(z)-f(z_o))}g(z)\;dz\right| &\leq \int_{\mathbb{R}\setminus[a-z_o,b-z_o]} e^{-T(f(z_o+x)-f(z_o))}\,|g(z_o+x)|\;dx\\ &\leq G_o\int_{\mathbb{R}\setminus[a-z_o,b-z_o]} e^{-T(\mu-\lambda)|x|}\;dx\in o\left(T^{-1}\right)\end{aligned} \tag{2.42}$$

as $T\to\infty$.

□

References

[1] *Hinch, E.J. (1991)*, Perturbation Methods
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