Pod systems: an equivariant ordinary differential equation approach to dynamical systems on a spatial domain

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Abstract

We present a class of systems of ordinary differential equations (ODEs), which we call ‘pod systems’, that offers a new perspective on dynamical systems defined on a spatial domain. Such systems are typically studied as partial differential equations, but pod systems bring the analytic techniques of ODE theory to bear on the problems, and are thus able to study behaviours and bifurcations that are not easily accessible to the standard methods. In particular, pod systems are specifically designed to study spatial dynamical systems that exhibit multi-modal solutions.

A pod system is essentially a linear combination of parametrized functions in which the coefficients and parameters are variables whose dynamics are specified by a system of ODEs. That is, pod systems are concerned with the dynamics of functions of the form \( \Psi(s, t) = y_1(t)\phi(s, x_1(t)) + \cdots + y_N(t)\phi(s, x_N(t)) \), where \( s \in \mathbb{R}^n \) is the spatial variable and \( \phi : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R} \). The parameters \( x_i \in \mathbb{R}^d \) and coefficients \( y_i \in \mathbb{R} \) are dynamic variables which evolve according to some system of ODEs, \( \dot{x}_i = G_i(x, y) \) and \( \dot{y}_i = H_i(x, y) \), for \( i = 1, \ldots, N \). The dynamics of \( \Psi \) in function space can then be studied through the dynamics of the \( x \) and \( y \) in finite dimensions.

A vital feature of pod systems is that the ODEs that specify the dynamics of the \( x \) and \( y \) variables are not arbitrary; restrictions on \( G_i \) and \( H_i \) are required to guarantee that the dynamics of \( \Psi \) in function space are well defined (that is, that trajectories are unique). One important restriction is symmetry in the

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ODEs which arises because $\Psi$ is invariant under permutations of the indices of the $(x_i, y_j)$ pairs. However, this is not the whole story, and the primary goal of this paper is to determine the necessary structure of the ODEs explicitly to guarantee that the dynamics of $\Psi$ are well defined.

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1. Introduction

The purpose of this paper is to introduce a new set of analytic tools, called pod systems, for the study of certain aspects of dynamical systems on a spatial domain, and in particular of certain bifurcation problems for such systems. Pod systems are a generalization and extension of the ‘pods’ introduced in [2] and developed further in [12,21,22], and are (in this incarnation) based on the observation that it is possible to interpret the dependent variables of a system of ordinary differential equations (ODEs) as coefficients and parameters in a dynamic linear combination of parametrized functions. Of course, this is not possible in any meaningful sense for arbitrary systems of ODEs, and the defining property of pod systems is that this interpretation (via the construction that we detail below) yields well-defined dynamics in the space of functions over the given domain. This property implies certain constraints on the vector field underlying a pod system, including, but not limited to, $S_n$-equivariance. The main result of this paper is the derivation of this structure in theorem 2.3.

Dynamical systems on spatial domains are, of course, usually studied as partial differential equations (PDEs), and because PDE theory already encompasses a vast wealth of analytical and numerical tools with which to study spatial dynamical systems it may not be immediately clear why a new approach is necessary. Pod systems offer an approach that complements PDE theory, in that it gives a perspective that highlights certain features of spatial dynamics that are difficult to study with PDEs. In particular, pod systems are strongly motivated by questions arising about macroscopic biological systems, especially with reference to the behaviour of systems near to unimodal steady states. This contrasts significantly with PDE theory in which spatially homogeneous steady states play an important role, reflecting their emphasis on physical and chemical systems in which low-energy states occupy a privileged position.

The question of the origin of species is a particularly good motivating example. If a population of organisms is conceived of as a density distribution over phenotype space then a population consisting of a single morphological type corresponds to a unimodal distribution. A pressing question is, what happens, generically, when such a distribution loses stability as the result of changing environmental drivers? Answers have been varied, ranging from bi- or multimodal distributions (corresponding to the first step in the formation of new species [3–6]) to a continuum of morphological types (which is incompatible with speciation [19]). Much of the disagreement is due to the lack of a framework that can provide generic, model-independent insights; insights that are clearly needed because of the great diversity of ecological and evolutionary scenarios. Bifurcations from unimodal steady states are studied in detail in [8], and in [9] pod systems are applied specifically to the problem of the origin of species.

This paper is concerned with introducing pod systems and studying their basic properties. In section 2 we define pod systems and state our main result, theorem 2.3, which gives a general form for systems of ODEs satisfying the definition of a pod system. Then in section 3 we give two examples that illustrate some of their features and motivate further study. Section 4
examines the structure of pod systems in more detail and sketches out the proof of theorem 2.3, and in section 5 we study some of the consequences of this structure for solutions that may occur in pod systems. In section 4 we also state our second result, theorem 4.1, which gives a general form for equivariant mappings $R^N \times R^N \rightarrow R^N \times R^N$ that are equivariant under a particular action of $S_N$. The details of the proofs of theorems 2.3 and 4.1 are given in section 6.

2. The fundamentals of pod systems

For simplicity we restrict attention to the dynamics of functions defined over $R$, although there is no theoretical obstacle to generalizations to $R^n$. The basis of the pod approach is the linear combination

$$\Psi_{(x,y)}(s) \overset{\text{def}}{=} \sum_{i=1}^{N} y_i \phi(s; x_i),$$

(2.1)

where $s \in R$ is the spatial variable, $x_i \in R^d$, $y_i \in R$ and $\phi : R \times R^d \rightarrow R$ is a function over $R$ parametrized by $d$ parameters in the form of $x_i$. The $(x, y)$ subscript denotes that the linear combination is specified by the vectors

$$x = (x_1, \ldots, x_N) \in X = R^{Nd} \quad \text{and} \quad y = (y_1, \ldots, y_N) \in Y = R^N.$$

We call the vector $(x, y) \in X \times Y$ a pod vector and the function $\phi$ a pod function. The term “pod” is used to refer to either the weighted pod function $y_i \phi(\cdot; x_i)$ or to the pair $(x_i, y_i)$.

The idea behind pod systems is to specify the dynamics of the function $\psi_{(x,y)}(s)$ by making $x$ and $y$ dynamic variables, $x(t)$ and $y(t)$, where $t$ is time. That is, we want to express $\Psi(s, t)$ as a dynamic linear combination

$$\Psi_{(x(t),y(t))}(s) \overset{\text{def}}{=} \sum_{i=1}^{N} y_i(t) \phi(s; x_i(t)),$$

(2.2)

where the dynamics of $x$ and $y$ are given by

$$\frac{dx_i}{dt} = G_i(x, y) \quad \text{and} \quad \frac{dy_i}{dt} = H_i(x, y), \quad i = 1, \ldots, N$$

(2.3)

for $G_i : X \times Y \rightarrow R^d$ and $H_i : X \times Y \rightarrow R$. Thus, differentiating (2.2) with respect to $t$ yields

$$\frac{d}{dt} \Psi_{(x(t),y(t))}(s) = \sum_{i=1}^{N} H_i(x(t), y(t)) \phi(s; x_i(t)) + y_i \frac{\partial \phi(s; x_i)}{\partial x_i} \cdot G_i(x(t), y(t)).$$

(2.4)

We will sometimes use the more standard notation $\Psi(s, t)$ in place of $\Psi_{(x(t),y(t))}(s)$, when it is not necessary to explicitly emphasize that the time dependence of $\Psi$ comes from the pod vector $z(t) = (x(t), y(t))$. These notational conventions denote only a difference of emphasis, and both refer to the same mathematical object, given by (2.2).

Letting $G = (G_1, \ldots, G_N) : X \times Y \rightarrow X$ and $H = (H_1, \ldots, H_N) : X \times Y \rightarrow Y$ we can write (2.3) as

$$\dot{x} = G(x, y),$$

$$\dot{y} = H(x, y),$$

(2.5)

where a dot denotes the derivative with respect to time. For simplicity, letting $z = (x, y) \in X \times Y = Z$ and $F = (G, H) : Z \rightarrow Z$, this reduces to

$$\dot{z} = F(z).$$

(2.6)
Thus, as \( z(t) = (x(t), y(t)) \) traces out a trajectory in \( Z = X \times Y \), the linear combination \( \Psi(s, t) = \Psi_z(t)(s) \) traces out a corresponding trajectory given by (2.2) in function space. However, we must be careful to ensure that the trajectories in function space are unique, and this does not follow from the well-defined dynamics in \( Z \) given by (2.6).

The problem is that there is redundancy in the linear combination (2.1) in the sense that distinct pod vectors may yield identical functions (as we discuss further below and in section 4). Suppose that for some fixed \( t_0 \) two distinct pod vectors \( z(t_0) = (x(t_0), y(t_0)) \) and \( w(t_0) = (u(t_0), v(t_0)) \), \( z(t_0) \neq w(t_0) \), are such that \( \Psi_{z(t_0)} = \Psi_{w(t_0)} \). Then uniqueness of trajectories in function space requires that \( \Psi_{z(t)} = \Psi_{w(t)} \) for all \( t \). Unless \( G_i \) and \( H_i \) in (2.3) are appropriately constrained, this will in general not be the case. Explicitly, we require that \( F = (G, H) \) in (2.6) must be such that if

\[
\Psi_{z(t_0)} = \Psi_{w(t_0)}
\]

then

\[
\frac{\partial \Psi_{z(t)}(x)}{\partial t} \bigg|_{t=t_0} = \frac{\partial \Psi_{w(t)}(x)}{\partial t} \bigg|_{t=t_0}.
\]

We call (2.7) (that is, the implication ‘If (2.7a) then (2.7b)’) the uniqueness condition. The primary goal of this paper is to determine the necessary constraints on \( F \) explicitly to ensure that the uniqueness condition is satisfied.

As we will see, the uniqueness condition (2.7) imposes a great deal of structure on \( F \), and determining this structure is vital if we are to give pod systems substantial content. Giving explicit meanings to (2.7a) and (2.7b) in terms of \( x \) and \( y \) will be instrumental in determining this structure, and we begin with the following observation about the redundancy in the linear combination (2.1) that makes the occurrence of (2.7a) a possibility.

The redundancy in (2.1) arises, in part, because the pod functions \( \phi(\cdot; x_i), i = 1, \ldots, N \) are not required to be linearly independent. Linear dependence in the pod functions can occur in many ways. The simplest scenario is that if \( x_i = x_j \) for some \( i \neq j \) then \( \phi(\cdot; x_i) = \phi(\cdot; x_j) \). Note that this is true for any choice of \( \phi \). However, for certain choices of \( \phi \) it is possible to obtain linear dependence even if \( x_i \neq x_j \) for all \( i, j = 1, \ldots, N \). For example, if \( \phi(x; \cdot) \equiv const \) then the pod functions are linearly dependent for all \( x \). Of course there are many other examples, and each case entails different conditions on \( x \) in order for \( \{\phi(\cdot; x_i)\}_{i=1}^{N} \) to be linearly dependent. Thus, because we need to be explicit about the conditions on \( z = (x, y) \) and \( w = (u, v) \) such that \( \Psi_z = \Psi_w \) in (2.7a), we limit the choice of \( \phi \) to functions such that \( \{\phi(\cdot; x_i)\}_{i=1}^{N} \) is linearly dependent only if two or more of the \( x_i \)s coincide. That is, we require that \( \phi(\cdot; x) \) satisfies

\[
x_1, \ldots, x_N \text{ distinct } \implies \{\phi(\cdot; x_i)\}_{i=1}^{N} \text{ is linearly independent.} \tag{2.8}
\]

As one might expect from the second term in (2.4), explicitly determining when and how (2.7b) can occur entails similar linear dependence problems with the derivative \( \partial \phi(\cdot; x_i)/\partial x_i \). Thus we also require that

\[
x_1, \ldots, x_N \text{ distinct } \implies \left\{\frac{\partial \phi(\cdot; x_i)}{\partial x_i}\right\}_{i=1}^{N} \text{ is linearly independent.} \tag{2.9}
\]

Note that (2.8) and (2.9) are requirements we have imposed on the choice of \( \phi \) to make the uniqueness condition (2.7) tractable and enable us to get a handle on its consequences. They are not themselves consequences of (2.7).

The uniqueness condition (2.7) is fundamental to pod systems, for without it the dynamics of the linear combination (2.2) would not be well defined and we would be unable to say anything useful about dynamics in function space. Thus we make the following definition.
Definition 2.1. A pod system over \( R \) is a pair \((F, \phi)\) where \( \phi : R \times R^d \rightarrow R \) is a \( d \)-parameter family of functions over \( R \) satisfying the linear independence conditions (2.8) and (2.9), and \( F = (G, H) : X \times Y \rightarrow X \times Y \) is such that the uniqueness condition (2.7) is satisfied.

The most straightforward consequence of this definition is symmetry. Let \( S_N \) act on \( Z = X \times Y \) by simultaneously permuting the indices of the \((x_i, y_i)\) pairs:

\[
\gamma z = (\gamma x, \gamma y) = \left( x_{r-1}, \ldots, x_{r-1-N}, y_{r-1}, \ldots, y_{r-1-N} \right).
\]

(2.10)

Then we have the following lemma, which we prove in section 4.

Lemma 2.2. Let \((F, \phi)\) be a pod system over \( R \). Then \( F = (G, H) \) is \( S_N \)-equivariant under the action of \( S_N \) in (2.10). That is,

\[
F(\gamma z) = \begin{pmatrix} G(\gamma x, \gamma y) \\ H(\gamma x, \gamma y) \end{pmatrix} = \gamma \begin{pmatrix} G(x, y) \\ H(x, y) \end{pmatrix} = \gamma F(z)
\]

(2.11)

for all \( z = (x, y) \in Z \) and all \( \gamma \in S_N \).

(Note that we are abusing the notation in (2.10) and (2.11) by letting \( \gamma \) denote the action of an element of \( S_N \) on \( Z \) and on \( X \) and \( Y \). The meaning should be clear from context, so we will not complicate the notation to resolve the ambiguity.) This symmetry essentially arises because the labels assigned to the pods (i.e. the indices of the \((x_i, y_i)\) pairs) are arbitrary, so permuting them should not affect the dynamics. Although this is perhaps intuitive, we provide a proof of lemma 2.2 in section 4 because things are not as straightforward as they first appear, and require explicit use of the linear independence assumptions (2.8) and (2.9) on \( \phi \). In section 4 we also examine the further consequences of definition 2.1, and in particular the uniqueness condition (2.7), in more detail.

Accounting for all the consequences of the uniqueness condition (2.7) for \( F \) yields the central result of this paper, theorem 2.3, which gives an explicit form for \( F \) that satisfies the requirements of definition 2.1 in the case \( d = 1 \). Theorem 2.3 is proved in detail in section 6, but in section 4 we sketch out the central issues that need to be resolved in the proof. To simplify notation we define

\[
\begin{pmatrix} x_1^i y_1^i \\ x_2^i y_2^i \\ \vdots \\ x_N^i y_N^i \end{pmatrix} = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_N^i \end{pmatrix} \begin{pmatrix} y_1^i \\ y_2^i \\ \vdots \\ y_N^i \end{pmatrix}.
\]

Theorem 2.3. Suppose that \((F, \phi)\) is a pod system over \( R \) such that \( \phi : R \times R \rightarrow R \) satisfies (2.8) and (2.9), and \( F = (G, H) \) is a smooth mapping on \( X \times Y \). Then \( G \) and \( H \) may be written in the form

\[
G(x, y) = \sum_{i=0}^{\infty} g_i(\{\alpha_k\}_{k=0}^{\infty}) \left[ x_i^1 \right],
\]

(2.12a)

\[
H(x, y) = \sum_{i=0}^{\infty} h_i(\{\alpha_k\}_{k=0}^{\infty}) \left[ y_i^1 \right],
\]

(2.12b)

where

\[
\alpha_k = \alpha_k(x, y) = \sum_{m=1}^{N} x_m^k y_m
\]

(2.13)

and the \( g_i \) and \( h_i \) are \( C^\infty \) functions of the \( \alpha_k \).
We call (2.12) a pre-normal form for pod systems over $\mathbb{R}$, by which we mean that it is a canonical form of sorts, but does not incorporate any changes of coordinates or genericity assumptions.

**Remarks on theorem 2.3**

Another way of looking at theorem 2.3 is that if $F = (G, H)$ has the form (2.12) and $\phi$ satisfies (2.8) and (2.9) then the dynamic linear combination (2.2) specifies a well-defined trajectory (in the sense of (2.7)) in function space. The *raison d’être* of theorem 2.3 is that it makes possible the study of generic features of pod systems, and as previously mentioned, of generic bifurcations from unimodal steady-states.

This raises an important question: to what extent can we extrapolate results derived from pod systems to PDEs? More generally, we might ask: is there a ‘Pod-PDE Correspondence Theorem’ telling us when a given pod system corresponds to a particular PDE, and vice versa? We do not have such a theorem at this point. Indeed, it is not immediately clear what exactly would be meant by ‘correspondence’. However, in [9] we show explicitly how it is possible to derive a pod system from a given PDE so that the steady states, bifurcation points, and nonlinear degeneracies of the original PDE are preserved (although the transient dynamics are a little different). But it is important to emphasize that theorem 2.3 and definition 2.1 do not make any mention of an ‘underlying PDE’. Pod systems should therefore not be thought of as approximations to PDEs, as in Galerkin methods [14] or inertial manifold theory [10]. Pod systems can be seen as an alternative type of model, complementing PDEs but also pointing in new directions.

Another important feature of theorem 2.3 is that it places no restrictions on $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ beyond the specification of its domain (corresponding to fixing the dimension of the $x_i, d = 1$) and requiring that it satisfy (2.8) and (2.9). This means that a particular $F$ yields different well-defined trajectories in function space when used in conjunction with different $\phi$. This gives an additional facet to the usual ‘model independence’ of equivariant systems. For example, a model-independent conclusion in equivariant bifurcation theory may provide the symmetries of generically existing solution branches, but not their precise form or stability which depends on the details of the equations. Similarly for pod systems, how a generic, model-independent conclusion manifests in a particular system depends not only on the specific $F$ but also on the choice of $\phi$.

**3. Examples of pod systems**

As an example, consider the two-pod system

\[
G_i(x, y) = -3x_i - 2x_2 - 7x_1x_0 + 4a_3 + (8 + 3a_0^2 - 2a_1)x_i - (5 + 3a_0 + 4a_1)x_i^2 - 10x_i^3, \\
H_i(x, y) = (12 + 15a_0 - 2a_0^3 - 10x_1 - a_0x_i + 10x_i^3)y_i,
\]

for $i = 1, 2$, where the $a_k$ are given by (2.13). Figures 1 and 2 show the results of numerical simulations of (3.1). Figure 1 shows various two-dimensional projections of the four-dimensional chaotic attractor, and figure 2 shows the time series for the $x$ and $y$ variables, along with a density plot of $\Psi(s, t) = \Psi(x_1(t), y_1(t))(s)$. For the density plot we take $\phi(s, x_i)$ to be a Gaussian function with the mode occurring at $s = x_i$, although of course we could have chosen any function satisfying (2.8) and (2.9).
Figure 1. Projections of an apparently chaotic trajectory obtained from numerical simulation of (3.1).

An important feature of the system in (3.1) is that the dynamics of $\Psi(s, t)$ remain the same for any $N > 2$. Observe that, in general, if there exists a solution to a two-pod system

$$x = (x_1(t), x_2(t)), \quad y = (y_1(t), y_2(t))$$

then for any $N > 2$ we can construct solutions

$$u(t) = (x_1(t), \ldots, x_i(t), x_1(t), \ldots, x_{p+1}(t), \ldots, x_{N+(q-p)-1}(t)),$$

$$v(t) = (v_1(t), \ldots, v_N(t))$$

for any $p, q, 1 \leq p \leq N - 1, p + 1 \leq q \leq N$, where

$$v_1(t) + \cdots + v_p(t) = y_1(t), \quad v_{p+1}(t) + \cdots + v_q(t) = y_2(t),$$

$$v_{p+1}(t) = \cdots = v_N(t) = 0.$$  

(In fact, it is not necessary for $v_{p+1}(t) = \cdots = v_N(t) = 0$ if some of the $u_i(t), i > q$, are synchronized, but to avoid notational complications we shall not pursue this at this point.) Then it is straightforward to compute

$$G_1(u, v) = \cdots = G_p(u, v) = G_1(x, y), \quad G_{p+1}(u, v) = \cdots = G_q(u, v) = G_2(x, y),$$

$$\sum_{i=1}^p H_i(u, v) = H_1(x, y), \quad \sum_{i=p+1}^q H_i(u, v) = H_2(x, y), \quad \sum_{i=q+1}^N H_i(u, v) = 0.$$
Figure 2. Time series of $x(t)$ and $y(t)$ and a density plot of $\Psi(s, t) = \Psi_{(x(t), y(t))}(s)$, taking the pod functions $\phi$ to be Gaussian, for the trajectory shown in figure 1. When seen in terms of the dynamics of the function $\Psi(s, t)$, the density plot above depicts infinite-dimensional chaos. In the pod framework the geometry of the attractor is quite evident (as in figure 1), and the system can be studied in four dimensions.

and hence

$$\frac{\partial}{\partial t} \Psi_{(u(t), v(t))}(s) = \sum_{i=1}^{N} H_i(u, v) \phi(s; u_i) + v_i \frac{\partial \phi(s; u_i)}{\partial u_i} G_i(u, v)$$

$$= \sum_{i=1}^{2} H_i(x, y) \phi(s; x_i) + y_i \frac{\partial \phi(s; x_i)}{\partial x_i} G_i(x, y)$$

$$= \frac{\partial}{\partial t} \Psi_{(x(t), y(t))}(s).$$

So $\Psi_{(u(t), v(t))} = \Psi_{(x(t), y(t))}$ for all $t$.

Solutions such as (3.2), where the $x$ variables synchronize, are completely expected in $S_N$-equivariant systems because of the existence of flow-invariant fixed-point subspaces, which we discuss in section 4. Matters are slightly different in pod systems because of the lack of synchronization in the $y$ variables; observe that only the sums of the $v_i$'s and the $H_i(u, v)$'s are relevant, not their individual values. This leads us to introduce extended fixed-point subspaces in section 4.

For pod systems given by (3.1) with $N > 2$, we would expect additional solutions to exist where the $x$ variables desynchronize in various ways, and possibly for the stability of solutions such as (3.2) that exist for lower-dimensional systems to change (although this does not occur for (3.1)). In general, therefore, we would anticipate that applications take $N$ large to allow for a larger range of behaviours, and to ensure that the dynamics of $\Psi$ results from $F$ and not from the artificial restrictions on the number of pods. For example, if $\phi$ is unimodal and $N = 2$ then
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Figure 3. Time series of $x(t)$ and $y(t)$ and a density plot of $\Psi(s, t) = \Psi(x(t), y(t))(s)$, taking the pod functions $\phi$ to be Gaussian and $F = (G, H)$ as in (3.3) with $N = 50$.

for any $F$, $\Psi(s)$ can exhibit only uni- or bi-modal behaviour, and without considering larger numbers of pods it is not clear whether this behaviour is the result of restricting $N$ or a robust feature of the dynamics.

An interesting twist on this is shown by the following example. Consider the $N$-pod system

\[
G_i(x, y) = -\frac{\lambda}{(x_i^2 - \lambda)^2} \sum_{j=1}^{N} y_j e^{-(x_i - x_j)^2} \left((x_i^2 - \lambda)(x_i - x_j) + x_j\right) - x_i^3,
\]

\[
H_i(x, y) = \left(1 + x_i + \frac{\lambda}{x_i^2 - \lambda} \sum_{j=1}^{N} y_j e^{-(x_i - x_j)^2}\right) y_i.
\]

Time series plots of the $x$ and $y$ variables and a density plot of $\Psi(s, t)$ for Gaussian $\phi$ are shown in figure 3. It can be shown that for each $p = 1, \ldots, N$ there exists an equilibrium

\[
x_p = (0, \ldots, 0, \eta, \ldots, \eta),
\]

\[
y_p = (v_1, \ldots, v_p, v_{p+1}, \ldots, v_N)
\]

for some $\eta \in \mathbb{R}$, where

\[
\sum_{i=1}^{p} v_i = 1, \quad \sum_{i=p+1}^{N} v_i = 0.
\]
Observe that all of these equilibria yield identical $\Psi$:

$$\Psi_{(x_p, y_p)}(s) = \sum_{i=1}^{p} v_i \phi(s; 0) + \sum_{i=p+1}^{N} v_i \phi(s; \eta) = \tilde{\Psi}(s; 0) \quad \text{for all } p = 1, \ldots, N.$$  

However, as indicated in figure 3, only $(x_1, y_1)$ is stable. The reason for this is, loosely speaking, that the unstable eigenspace breaks the symmetry of the first $p$ components of $x_p$. When $p = 1$, however, this eigenspace does not exist, and hence there is no instability. Thus the stability of $(x_1, y_1)$ is an artefact of a finite number of pods, and it is reasonable to suppose that, given an infinite number of pods, the dynamics observed in figure 3 between $t = 100$ and $t = 150$ would continue indefinitely. In general, behaviours resulting from restrictions on the number of pods should be treated with caution, and for this reason, we consider infinite pod systems and some of the behaviours they can exhibit in more detail in section 5.

4. Symmetry and structure in pod systems

In this section we consider the structure that (2.7) imposes on $F$, and break the proof of theorem 2.3 down into manageable pieces. First, we prove lemma 2.2, showing that $F$ must be $S_N$-equivariant. Then in theorem 4.1 we give a pre-normal form for mappings on $X \times Y = \mathbb{R}^N \times \mathbb{R}^N$ that are equivariant with respect to the action of $S_N$ given in (2.10). The proof of theorem 2.3 consists primarily of refining this equivariant pre-normal form to take further constraints into account that arise from the uniqueness condition (2.7). Over the course of this discussion we require a few group-theoretic concepts, which we consider only insofar as they relate to pod systems. A more complete presentation of symmetric dynamics can be found in [11,12].

We begin with the proof of lemma 2.2.

**Proof of lemma 2.2.** Let $S_N$ act on $Z$ as in (2.10) and observe that $\Psi_{z} = \Psi_{\gamma z}$ for all $z = (x, y) \in Z, \gamma \in S_N$ since permuting the pod indices only changes the order of the summation in the linear combination (2.2). Thus, in order to satisfy (2.7), $F$ must be such that

$$\frac{\partial \Psi_z}{\partial t} = \frac{\partial \Psi_{\gamma z}}{\partial t} \quad \text{for all } \gamma \in S_N, z \in Z.$$

In general, for any $w = (u, v) \in X \times Y$

$$\frac{\partial \Psi_{w}}{\partial t}(s) = \sum_{i=1}^{N} H_i(w) \frac{\partial \phi(s; u_i)}{\partial u_i} \cdot G_{i}(w).$$

So if $w = \gamma z$ then

$$\frac{\partial \Psi_{\gamma z}}{\partial t}(s) = \sum_{i=1}^{N} H_i(\gamma z) \frac{\partial \phi(s; x_{\gamma^{-1} i})}{\partial x_{\gamma^{-1} i}} \cdot G_i(\gamma z). \quad (4.1)$$

Permuting the order of the summation in (4.1) by letting $i = \gamma j$ yields

$$\frac{\partial \Psi_{\gamma z}}{\partial t}(s) = \sum_{j=1}^{N} H_{\gamma j}(\gamma z) \frac{\partial \phi(s; x_j)}{\partial x_j} \cdot G_{\gamma j}(\gamma z). \quad (4.2)$$

But observe that

$$\frac{\partial \Psi_z}{\partial t}(s) = \sum_{j=1}^{N} H_{j}(z) \frac{\partial \phi(s; x_j)}{\partial x_j} \cdot G_{j}(z). \quad (4.3)$$
Equating (4.2) and (4.3), and assuming that $G$ and $H$ are independent yields

$$
\sum_{j=1}^{N} (H_{ij} (y z) - H_j (z)) \phi(s; x_j) = 0,
$$

$$
\sum_{j=1}^{N} y_j \frac{\partial \phi(s; x_j)}{\partial x_j} \cdot (G_{ij} (y z) - G_j (z)) = 0.
$$

Suppose that $z = (x, y)$ is such that $x_i \neq x_j$ for all $i \neq j$ and $y_i \neq 0$ for all $i$. Then the linear independence conditions (2.8) and (2.9) on $\phi$ imply that

$$
G_{ij} (y z) = G_j (z)
$$

and

$$
H_{ij} (y z) = H_j (z)
$$

for all $j = 1, \ldots, N$ and all $\gamma \in S_N$. Therefore,

$$
\gamma^{-1} G(y z) = G(z)
$$

and

$$
\gamma^{-1} H(y z) = H(z)
$$

for all $\gamma \in S_N$, which is precisely the equivariance condition (2.11). \hfill \Box

In section 6 we prove the following equivariant pre-normal form theorem for the case $d = 1$.

**Theorem 4.1.** Let $F = (G, H) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ be a smooth $S_N$-equivariant mapping in $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ under the action given by (2.10). Then $G$ and $H$ may be written in the form

$$
G(x, y) = \sum_{i,j=0}^{N-1} g_{ij} (\{\alpha_{kl}\}_{k,l=0}^{N}) \left[ x_i^1 y_j^1 \right],
$$

$$
H(x, y) = \sum_{i,j=0}^{N-1} h_{ij} (\{\alpha_{kl}\}_{k,l=0}^{N}) \left[ x_i^1 y_j^1 \right],
$$

where

$$
\alpha_{kl} = \alpha_{kl}(x, y) = \sum_{m=1}^{N} x_m^k y_m^l
$$

generate the $S_N$-invariants and $g_{ij}$ and $h_{ij}$ are $C^\infty$ functions in the invariants.

Note that the limits on the indices in (4.4) differ from those in (2.12); with $i$ and $k$ in (2.12) running up to infinity. This is because, in proving theorem 2.3 we must remove some of the restrictions placed on the invariants and equivariants in (4.4). For completeness, we leave theorem 4.1 in its stronger form. We discuss this further in section 6.

We now turn to the general properties $S_N$-equivariant vector fields, and look at what they can tell us about pod systems when seen in conjunction with the uniqueness condition (2.7). One important property of equivariant systems is that they possess flow-invariant subspaces, called fixed-point subspaces. The isotropy subgroup of a point $z \in Z$ is the subgroup

$$
\Sigma_z = \{ \gamma \in S_N : \gamma z = z \}.
$$

Then the fixed-point subspace of an isotropy subgroup $\Sigma$ is

$$
\text{Fix}(\Sigma) = \{ z \in Z : y z = z \text{ for all } \gamma \in \Sigma \}.
$$

Up to conjugacy the isotropy subgroups of $S_N$ have the form

$$
\Sigma = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_m}, \quad \text{where} \quad p_1 + p_2 + \cdots + p_m = N
$$

where $p_1, p_2, \ldots, p_m$ are the indices of the subgroup.
(see [12]). By (4.6) and the action of $\Sigma_N$ given in (2.10), if $\Sigma \subseteq \Sigma_N$ is an isotropy subgroup and $z = (x, y) \in \text{Fix}(\Sigma)$ then $x$ and $y$ have the form

$$x = (\tilde{x}_1, \ldots, \tilde{x}_1; \ldots; \tilde{x}_m, \ldots, \tilde{x}_m) \quad \text{and} \quad y = (v_1, \ldots, v_1; \ldots; v_m, \ldots, v_m)$$

(7)

where $\tilde{x}_i \in R^d$ and $v_i \in R$.

It is straightforward to show that fixed-point subspaces are flow-invariant, i.e. if $z \in \text{Fix}(\Sigma)$ then $F(z) \in \text{Fix}(\Sigma)$. In virtue of their symmetry, pod systems possess the standard flow-invariant fixed-point subspaces, but the uniqueness condition (2.7) implies the existence of larger flow-invariant subspaces that are not purely the result of symmetry, and which contain the standard fixed-point subspaces. These subspaces are obtained by restricting the action of $\Sigma_N$ to $X$. Explicitly, if $\Sigma \subseteq \Sigma_N$ is an isotropy subgroup of the form (4.6) then we define

$$\text{Fix}(\Sigma|_X) = \{x \in X : \gamma x = x \text{ for all } \gamma \in \Sigma\}$$

(8)

and similarly for $\text{Fix}(\Sigma|_Y)$. We shall refer to the subspace $\text{Fix}(\Sigma|_X) \times Y$ as the extended fixed-point subspace of $\Sigma$. Elements of $\text{Fix}(\Sigma|_X) \times Y$ thus have the form

$$(x, y) = (\tilde{x}_1, \ldots, \tilde{x}_1; \ldots; \tilde{x}_m, \ldots, \tilde{x}_m; y_1, \ldots, y_N) \quad \tilde{x}_i \in R^d, \quad y_i \in R.$$  

(9)

To simplify the linear combination (2.2) when $x \in \text{Fix}(\Sigma|_X)$ we define

$$q_0 = 0, \quad q_j = p_1 + \cdots + p_j, \quad q_m = N$$

(10)

and

$$\tilde{y}_j = \sum_{i=q_{j-1}+1}^{q_j} y_i.$$  

(11)

for $j = 1, \ldots, m$. Then if $z = (x, y) \in \text{Fix}(\Sigma|_X) \times Y$

$$\Psi_z(x) = \sum_{j=1}^{m} \tilde{y}_j \phi(x; \tilde{x}_j).$$

(12)

**Lemma 4.2.** If $(F, \phi)$ is a pod system and $\Sigma$ is an isotropy subgroup of $\Sigma_N$ then $\text{Fix}(\Sigma|_X) \times Y$ is flow-invariant for $F$.

**Proof.** Suppose that $z = (x, y) \in \text{Fix}(\Sigma|_X) \times Y$ where $\Sigma$ is an isotropy subgroup of the form (4.6), and let

$$w = (\tilde{y}_1, \ldots, \tilde{y}_1; \ldots; \tilde{y}_m, \ldots, \tilde{y}_m) \in \text{Fix}(\Sigma|_Y)$$

(13)

so that

$$\sum_{i=q_{j-1}+1}^{q_j} w_i = \sum_{i=q_{j-1}+1}^{q_j} y_i$$

for all $j = 1, \ldots, m$. We show that for any pod system

$$G(x, y) = G(x, w).$$

(14)

Then because $(x, w) \in \text{Fix}(\Sigma)$, it follows that $F(x, w) \in \text{Fix}(\Sigma)$, and hence, that $G(x, w) \in \text{Fix}(\Sigma|_X)$. Equation (4.14) therefore implies that $G(x, y) \in \text{Fix}(\Sigma|_X)$, and hence that $F(x, y) \in \text{Fix}(\Sigma|_X) \times Y$. 


Observe that we can write any \( y \in Y \) uniquely as \( y = w + v \) where \( w \) is as in (4.13) and \( v \in \mathrm{Fix}(\Sigma_Y)\) so that

\[
\sum_{i=q_{j-1}+1}^{q_j} v_i = 0 \quad \text{for all } j = 1, \ldots, m.
\]

It is straightforward to verify that \( \Psi_{(x,w)} = \Psi_{(x,w+v)} \). Then \( \partial_i \Psi_{(x,w)} = \partial_i \Psi_{(x,w+v)} \) if and only if

\[
\sum_{i=1}^{N} H_i(x, w)\phi(s; x_i) + w_i \frac{\partial \phi(s; x_i)}{\partial x_i} G_i(x, w)
\]

\[
= \sum_{i=1}^{N} H_i(x, w+v)\phi(s; x_i) + (w_i + v_i) \frac{\partial \phi(s; x_i)}{\partial x_i} G_i(x, w+v).
\]

Assuming that \( G \) and \( H \) are independent, and recalling the linear independence conditions (2.8) and (2.9), this entails that for each \( j = 1, \ldots, m \)

\[
\sum_{i=q_{j-1}+1}^{q_j} (w_i G_i(x, w) - (w_i + v_i) G_i(x, w + v)) = 0 \tag{4.15}
\]

(\( q \) is a similar equation for \( H \) which does not concern us at present).

Observe that for \( j = 1, \ldots, m \)

\[
\sum_{i=q_{j-1}+1}^{q_j} w_i G_i(x, w) = \sum_{i=q_{j-1}+1}^{q_j} (w_i + v_i) G_i(x, w)
\]

because \( G(x, w) \in \mathrm{Fix}(\Sigma_X) \) and \( v \in \mathrm{Fix}(\Sigma_Y) \). Thus, equation (4.15) can be rewritten as

\[
\sum_{i=q_{j-1}+1}^{q_j} (w_i + v_i)(G_i(x, w) - G_i(x, w + v)) = 0.
\]

So \( G_i(x, w) = G_i(x, w + v) \) for all \( i = 1, \ldots, N \) since the uniqueness condition requires that this holds for all \( y = w + v \in R^N \). Thus equation (4.14) is verified, and the result follows.

These extended fixed-point subspaces will not occur for the general \( S_N \)-equivariant pre-normal form of theorem 4.1 because \( S_N \)-equivariant systems will not, in general, satisfy the uniqueness condition (2.7). By definition 2.1, however, such subspaces must occur in all pod systems.

To complicate matters further, different isotropy subgroups are, in a sense, dynamically indistinguishable. Consider for example two isotropy subgroups \( \Sigma = S_{p_1} \times S_{N-p_1} \) and \( \Delta = S_{p_2} \times S_{N-p_2} \), where \( p_1 \neq p_2 \). Let

\[
z = \left( \tilde{x}_1, \ldots, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_2; \frac{\tilde{y}_1}{p_1}, \ldots, \frac{\tilde{y}_1}{p_1}, \frac{\tilde{y}_2}{N-p_1}, \ldots, \frac{\tilde{y}_2}{N-p_1} \right) \in \mathrm{Fix}(\Sigma)
\]

and

\[
w = \left( \tilde{x}_1, \ldots, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_2; \frac{\tilde{y}_1}{p_2}, \ldots, \frac{\tilde{y}_1}{p_2}, \frac{\tilde{y}_2}{N-p_2}, \ldots, \frac{\tilde{y}_2}{N-p_2} \right) \in \mathrm{Fix}(\Delta)
\]
for some $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^d$ and $\tilde{y}_1, \tilde{y}_2 \in \mathbb{R}$. Then
\[
\Psi_i(x) = \tilde{y}_1\phi(s; \tilde{x}_1) + \tilde{y}_2\phi(s; \tilde{x}_2) = \Psi_o(s).
\]
So $F$ must be such that the uniqueness condition (2.7) holds for all such vectors $z$ and $w$, even though they lie in different fixed-point subspaces. Taking all of the above requirements into account, and refining the form of the general equivariant accordingly, as we do in section 6, enables us to prove theorem 2.3.

5. Solutions and equilibria in pod systems

Solutions to $(\dot{x}, \dot{y}) = F(x, y)$ that lie in the extended fixed-point subspace $\text{Fix} (\Sigma |_X) \times Y$ have the form given in (4.9). If $\phi$ is unimodal then the corresponding $\Psi_{(x,y)}$ is $m$-modal. In particular, if $(x, y) \in \text{Fix}(S_{\mathcal{N}}|_X) \times Y$ is fully symmetric in the $x$-variables, $x_1 = \cdots = x_N = \tilde{x}$, then $\Psi_{(x,y)}(s) = y\phi(s; \tilde{x})$, where $y = y_1 + \cdots + y_N$. Thus if $\phi$ is unimodal then a fully symmetric is also unimodal. This ‘fully symmetric’ in the context of pod systems means something very different from ‘fully symmetric’ in the context of PDEs, where the symmetry is given by the action of a Euclidean group and full symmetry implies spatial homogeneity.

While on the subject of spatially homogeneous solutions, note that for any choice of $\phi$, if $y = 0$ then $\Psi_{(x,0)}(s) = 0$. Furthermore, using the pre-normal form for $H$ in (2.12b), $H(x, 0) \equiv 0$, so these spatially homogeneous solutions are steady states for the dynamics of $\Psi_{(x(t),y(t))}(s)$ (though $G(x, 0)$ is not necessarily zero, so $(x, 0)$ need not be an equilibrium for $F$). Furthermore, observe that (2.8) prohibits $\phi$ being a constant over the domain, so obtaining spatially homogeneous states for nonzero $y$ is problematic for pod systems. Thus, while it is notable that such spatially homogeneous steady states naturally exist in all pod systems, it is not clear that pod systems would add much to the standard PDE theory, so we will not pursue this point further here.

In section 3 we noted that we must be wary of behaviours arising because of a limited number of pods in a system. Moreover, it is reasonable to suppose that systems consisting of a large number of finely grained pod functions allow for greater resolution and the ability to capture more dynamic phenomena. With this in mind, we turn our attention to pod systems consisting of an infinite number of pods.

In order to study infinite pod systems we consider a slightly specialized class of pod functions. To begins with, we restrict the spatial domain to $\mathbb{R}$ and fix $d = 1$ so that $x \in \mathbb{R}^N$. In many applications it is natural to let $x_i$ specify the location of the $i$th pod over $\mathbb{R}$ (for example, as the mode of a unimodal function). In these cases $\phi(s; x_i) = \phi(s - x_i)$. Suppose in addition that $\phi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

\[
\begin{align*}
(a) & \quad \phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \\
(b) & \quad \int_{-\infty}^{\infty} \phi(s) \, ds = 1
\end{align*}
\]

so that in particular $\phi(s) \rightarrow 0$ as $s \rightarrow \pm \infty$. Define the dilation $\phi_n$ by
\[
\phi_n(s) = \phi(ns) \quad s \in \mathbb{R}, \quad n > 0.
\]

Then
\[
\int_{-\infty}^{\infty} \phi_n(s) \, ds = \frac{1}{n}.
\]
For example, letting \( \phi(s) \) be a standard normal distribution clearly satisfies (5.1), and the
dilation is then
\[
\phi_n(s - x_i) = \frac{1}{n} \frac{e^{-n\sigma^2 s^2}}{\sqrt{2\pi}\sigma},
\]
where \( \sigma = 1/n \).

In proposition 5.1 we show that for any \( q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^0(\mathbb{R}) \) and any \( \epsilon > 0 \)
we can, given large enough \( n \) and \( N \), construct a pod vector \( w_0 \) such that \( \|q - \Psi_{w_0}\|_2 < \epsilon \), where
\( \| \cdot \|_2 \) is the \( L^2 \) norm. In corollary 5.2 we show that if \( \phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^\infty \) then there
exists a pod system with equilibrium \( w_0 \) such that \( \|q - \Psi_{w_0}\|_2 < \epsilon \).

While a corresponding result in larger spaces would be desirable, for our purposes in
ecology and evolution (where we are typically dealing with density distributions) the space \( L^2 \) is
usually sufficient. Thus, while the power of pod systems may be limited when studying spatially
homogeneous states, they are particularly well-suited to dealing with more biologically relevant
spatial dynamics. The requirement that \( \phi \in C^\infty \) is to ensure that a pod dynamic can be
constructed such that the \( g_1 \) and \( h_1 \) of theorem 2.3 are also smooth. These requirements stem
from Schwartz’s theorem \cite{1, 15, 20} and Poénaru’s theorem \cite{11, 18} which are used in the
derivation of the equivariant pre-normal form of theorem 4.1.

Let \( q : \mathbb{R} \to \mathbb{R}, n, k \in \mathbb{N}, \) and take \( N = 2nk + 1 \). Renumbering the subscripts, let
\[
\Psi_{w_0}(s) = \sum_{j=-nk}^{nk} v_j \phi_n(s - u_j),
\]
then
\[
\Psi_{w_0}(s) = \sum_{j=-nk}^{nk} v_j \phi_n(s - u_j),
\]
\( \Psi_{w_0}(s) = \sum_{j=-nk}^{nk} v_j \phi_n(s - u_j). \)

\[ \textbf{Proposition 5.1.} \text{ Let } \phi \text{ be a pod function satisfying (5.1) and let } q : \mathbb{R} \to \mathbb{R}, \text{ where } q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^0(\mathbb{R}). \text{ Then for all } \epsilon > 0 \text{ there exist } n, k \in \mathbb{N} \text{ such that for } \Psi_{w_0} \text{ given by (5.4)} \]
\[ \|q - \Psi_{w_0}\|_2 < \epsilon \]
where \( \| \cdot \|_2 \) is the \( L^2 \) norm.

\[ \textbf{Proof.} \text{ The proof is a simple application of the Fourier transform. Recall that if } q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ then the Fourier transform } \hat{q} \text{ of } q \text{ is} \]
\[ \hat{q}(\omega) = \int_{-\infty}^{\infty} q(s) e^{-i\omega s} ds. \]
When \( q \in C^0(\mathbb{R}) \) we can express (5.5) as the limit of Riemann sums:
\[ \hat{q}(\omega) = \lim_{k \to \infty} \lim_{n \to \infty} \sum_{j=-nk}^{nk} q(s) \frac{j}{n} e^{-i\omega j/n}. \]
We show that the right-hand side of (5.6) is the Fourier transform of
\[ p(s) = \lim_{k \to \infty} \lim_{n \to \infty} \sum_{j=-nk}^{nk} q(s) \frac{j}{n} \phi_n(s - j/n), \]
where $\phi_n$ is the dilation (5.2). This follows by standard properties of the Fourier transform (see for example Mallat [13, p 25]. Namely, if $q(s) \mapsto \hat{q}(\omega)$ then $q(s - a) \mapsto e^{-ia\omega} \hat{q}(\omega)$ and $q(bs) \mapsto \frac{1}{|b|} \hat{q}(\omega/b)$.

Compute

$$
\hat{p}(\omega) = \lim_{k \to \infty} \lim_{n \to \infty} \sum_{j=\neg nk}^{nk} q\left(\frac{j}{n}\right) \phi_n\left(s - \frac{j}{n}\right)
= \lim_{k \to \infty} \lim_{n \to \infty} \sum_{j=\neg nk}^{nk} q\left(\frac{j}{n}\right) e^{-i\omega j/n} \phi\left(\frac{\omega}{n}\right)
= \lim_{k \to \infty} \lim_{n \to \infty} \hat{\phi}\left(\frac{\omega}{n}\right) \sum_{j=\neg nk}^{nk} q\left(\frac{j}{n}\right) e^{-i\omega j/n}.
$$

But by (5.1)

$$
\lim_{n \to \infty} \hat{\phi}\left(\frac{\omega}{n}\right) = \hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(s) \, ds = 1,
$$
so the result follows because the Fourier transform preserves the norm (up to a factor of $1/2\pi$), a result known as Parseval’s theorem (see for example Mallat [13, p 26].

**Corollary 5.2.** Let $q : \mathbb{R} \to \mathbb{R}$, where $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^0$. Then there exists a pod system with equilibrium $u_0$ such that $\|q - \Psi_{u_0}\|_2 < \epsilon$.

**Proof.** By proposition 5.1, given any $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^0$ and any $\epsilon > 0$, a pod vector $u_0$, given by (5.3), can be constructed so that $\|q - \Psi_{u_0}\|_2 < \epsilon$. Define $F = (G, H)$ by

$$
G_i(x, y) = \frac{\partial}{\partial x_i} \left[ \Psi_{u_0}(s) - \Psi_{(x, y)}(s) \right]_{s=x_i},
$$

$$
H_i(x, y) = \left( \Psi_{u_0}(x_i) - \Psi_{(x, y)}(x_i) \right) y_i,
$$
for $i = 1, \ldots, N = 2nk + 1$, where $n, k \in \mathbb{N}$ are as in proposition 5.1. Clearly, (5.8) has $(x, y) = u_0$ as a steady state. Clearly $F = (G, H)$ is equivariant under the action in (2.10). That it is a pod system also can be verified by the results used in the proof of theorem 2.3 in section 6.

6. Proof of theorem 2.3

In this section we again consider pod systems $(F, \phi)$ with $d = 1$, so that $x_i \in \mathbb{R}$ and $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$. First we prove theorem 4.1 giving the pre-normal form for $\mathbb{S}_N$-equivariants in $x$ and $y$. In fact, (4.4) of theorem 4.1 is too restrictive in the limits on the indices for reasons we discuss below. Removing these limits, and imposing the uniqueness condition (2.7) yields (2.12), proving theorem 2.3.

Theorem 4.1 is an application Poenaru’s theorem [11, 18], which is a corollary of Schwartz’s Theorem, [1, 15, 20]. Let $z = (x, y) = (x_1, \ldots, x_N; y_1, \ldots, y_N)$ and let $\mathbb{S}_N$ act on $\mathbb{R}^N \times \mathbb{R}^N$ as in (2.10). It is well known that the ring of $\mathbb{S}_N$-invariant polynomials in $x$ and $y$ is generated by the invariant monomials

$$
\alpha_{kl}(x, y) = \sum_{i=1}^{N} x_i^k y_i^l \quad \text{for} \quad k, l = 0, \ldots, N,
$$

(6.1)
(see, for example, [16, 17]). Then Poénaru’s theorem (adapted for our purposes) states that there exist \(S_N\)-equivariant polynomial maps \(E_1, \ldots, E_m : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N\) such that, for every \(C^\infty\) \(S_N\)-equivariant map \(F : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N\), there exist \(C^\infty\) functions \(p_1, \ldots, p_m : \mathbb{R} \rightarrow \mathbb{R}\) such that

\[
F(z) = \sum_{l=1}^m p_l([\alpha_{il}(x, y)]_{k,\ell=0}^N)E_l(z).
\]

Thus, proving theorem 4.1 consists of showing that the \([x^i_1y^j_1]\) of (6.5) are in fact the \(E_l\) in (6.2).

**Proposition 6.1.** The module of \(S_N\)-equivariant polynomial mappings \(P : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N\)
in \(x = (x_1, \ldots, x_N)\) and \(y = (y_1, \ldots, y_N)\) is generated over the \(S_N\)-invariants by

\[
E_{ij} = (x^i_1y^j_1, \ldots, x^i_Ny^j_N)^T = \left[ x^i_1y^j_1 \right]
\]

for \(i, j = 0, \ldots, N - 1\).

The proof of proposition 6.1 is given in appendix A. If \(F = (G, H) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N\) is \(S_N\)-equivariant under the action in (2.10) then both \(G\) and \(H\) may be written as

\[
\sum_{i,j=0}^{N-1} p_i([\alpha_{il}(x, y)]_{k,\ell=0}^N)\left[ x^i_1y^j_1 \right]
\]

and theorem 4.1 follows by Poénaru’s theorem. Good sources of general theoretical background for equivariant mappings are [11, chapter XII, sections 4–6] and (for one-variable \(S_N\)-equivariant mappings) [12, sections 2.4 and 2.6].

We prove theorem 2.3 by determining the \(S_N\)-equivariant mappings that also satisfy condition (2.7), and we do this by restricting the invariants, \(\alpha_{il}\), and the equivariants, \([x^i_1y^j_1]\), while \(g_{ij}\) and \(h_{ij}\) remain arbitrary smooth functions. However, because of restrictions already made in the proof of theorem 4.1 on the orders of the invariants and equivariants, this second restriction is too strong if applied directly to (4.4).

To illustrate the point, recall the example of (3.1) in section 3, and note that when \(N = 2\) we can rewrite \(x^3_i\) and \(a_{31}\) occurring in \(G_i(x, y)\) in terms of lower order invariants as

\[
x^3_i = \alpha_{10}x^2_i - \frac{1}{2}(\alpha_{10}^2 - \alpha_{20})x_i,
\]

\[
a_{31} = \alpha_{10}a_{21} - \frac{1}{2}(\alpha_{10}^2 - \alpha_{20})a_{11}.
\]

Thus (3.1) can be written in terms of lower order invariants and equivariants, but the expressions in (6.4) contain the invariants \(\alpha_{10}\) and \(\alpha_{20}\) which are absent from the pod system pre-normal form of (2.12). Clearly, if the substitution is made then (3.1) still satisfies the definition of a pod system, but it does not conform to the pre-normal form of (2.12).

In the proof of theorem 4.1, substitutions of the form (6.4) are used to reduce the order of the invariants and equivariants present in (4.4). But to take account of the additional invariants that arise (such as the \(\alpha_{10}\) and \(\alpha_{20}\) in (6.4)) complicates matters considerably. Therefore, instead of starting with the general equivariant mappings (6.5), we begin the proof of theorem 2.3 by allowing \(i, j, k\) and \(l\) in (4.4) to range up to infinity:

\[
G(x, y) = \sum_{i,j=0}^{\infty} g_{ij}(\alpha_{il})_{k,\ell=0}^\infty \left[ x^i_1y^j_1 \right],
\]

\[
H(x, y) = \sum_{i,j=0}^{\infty} h_{ij}(\alpha_{il})_{k,\ell=0}^\infty \left[ x^i_1y^j_1 \right],
\]
Much of this section is concerned with dynamics in the fixed-point subspaces of isotropy subgroups of \( S_N \), so we recall the notation introduced in section 4; in particular, (4.10), (4.11) and (4.12). By analogy with (4.11), we also define

\[
\tilde{y}_j^i = \sum_{k=q_{j-1}+1}^{q_j} y_k^i. \tag{6.6}
\]

Let \( \Sigma \) be an isotropy subgroup of \( S_N \) of the form (4.6) so that if \( x \in \text{Fix}(\Sigma|_X) \) then

\[
x = (\tilde{x}_1, \ldots, \tilde{x}_1, \ldots, \tilde{x}_m, \ldots, \tilde{x}_m). \tag{6.7}
\]

Recall from lemma 4.2 that \( \text{Fix}(\Sigma|_X) \times Y \) is flow-invariant for \( F \). By analogy with (6.7) we write \( G(z) \in \text{Fix}(\Sigma|_X) \) as

\[
G(z) = (\tilde{G}_1(z), \ldots, \tilde{G}_n(z)), \ldots, \tilde{G}_1(z), \ldots, \tilde{G}_n(z)), \tag{6.7}
\]

and by analogy with (4.11) we write

\[
\tilde{H}_j(z) = \sum_{k=q_{j-1}+1}^{q_j} H_k(z).
\]

Suppose that two-pod vectors \( z = (x, y) \) and \( w = (u, v) \) satisfy (2.7a) so that \( \Psi_z = \Psi_w \). We can assume without loss of generality that \( x \) and \( u \) lie in the fixed-point subspaces of the (possibly trivial) isotropy subgroups

\[
\Sigma = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_m}, \quad \Delta = S_{r_1} \times S_{r_2} \times \cdots \times S_{r_n} \tag{6.8}
\]

for some \( 1 \leq m, n \leq N \), where \( p_1 + \cdots + p_m = r_1 + \cdots + r_n = N \). We may also assume (again, without loss of generality) that for some \( \xi, 0 \leq \xi \leq \min\{m, n\}, \)

\[
\tilde{x}_i = \tilde{u}_i \quad \text{for all } i = 1, \ldots, \xi, \]

\[
\tilde{x}_i \neq \tilde{u}_j \quad \text{if } i > \xi \text{ or } j > \xi. \tag{6.9}
\]

Observe that isotropy subgroups \( \Sigma \) and \( \Delta \) are not conjugate and their fixed-point subspaces have dimension \( m \) and \( n \), respectively.

The proof of theorem 2.3 begins by determining the necessary and sufficient conditions on \( z \) and \( w \) for (2.7a) and (2.7b) to hold. These conditions are given by the following two lemmas. Although the main result concerns pod systems in which \( d = 1 \) (that is, where the pods come from a one-parameter family), lemmas 6.2 and 6.3 are completely general and apply to any pod system.

**Lemma 6.2.** Let \( \Sigma \) and \( \Delta \) be as in (6.8), and let \( z = (x, y) \) and \( w = (u, v) \) such that \( x \in \text{Fix}(\Sigma|_X) \) and \( u \in \text{Fix}(\Delta|_X) \) and (6.9) holds for some \( \xi \). Then \( \Psi_z = \Psi_w \) if and only if

\[
\tilde{y}_i = \tilde{v}_i \quad \text{for all } i = 1, \ldots, \xi, \]

\[
\tilde{y}_{\xi+1} = \cdots = \tilde{y}_m = \tilde{v}_{\xi+1} = \cdots = \tilde{v}_n = 0. \tag{6.10}
\]

**Proof.** Suppressing the \( s \) variable in \( \phi(s; x_i) \), observe that

\[
\Psi_z = \sum_{i=1}^{m} \tilde{y}_i \phi(\tilde{x}_i) = \sum_{i=1}^{n} \tilde{v}_i \phi(\tilde{u}_i) = \Psi_w
\]
if and only if
\[ \sum_{i=1}^{\xi} (\tilde{y}_i - \tilde{v}_i) \phi(x_i) + \sum_{i=\xi+1}^{m} \tilde{y}_i \phi(x_i) - \sum_{i=\xi+1}^{n} \tilde{v}_i \phi(u_i) = 0 \]  
(6.11)
by (6.9). By (2.8) and (6.9) the set
\[ \{ \phi(x_1), \ldots, \phi(x_\xi), \phi(x_{\xi+1}), \ldots, \phi(x_m), \phi(u_{\xi+1}), \ldots, \phi(u_n) \} \]
is linearly independent. Therefore, (6.11) holds if and only if \( y \) and \( v \) satisfy (6.10). \( \square \)

Note that by (6.6), the equalities in (6.10) do not imply that \( \tilde{y}_i^l = \tilde{v}_i^l \) or that \( \tilde{y}_i^{\xi+1} = \cdots = \tilde{v}_i^l = 0 \) for \( l > 1 \).

**Lemma 6.3.** Make the same assumptions as for lemma 6.2 and suppose that \( \Psi_{\xi} = \Psi_u \). Then \( \partial_i \Psi_{\xi} = \partial_i \Psi_u \) if and only if
\[ \tilde{G}_i(x, y) = \tilde{G}_i(u, v) \]  
(6.12)
and
\[ \tilde{H}_i(x, y) = \tilde{H}_i(u, v), \]  
(6.13)
for all \( i = 1, \ldots, \xi, \) and
\[ \tilde{H}_i(x, y) = 0 \quad \text{for } i = \xi + 1, \ldots, m \]
\[ \tilde{H}_i(u, v) = 0 \quad \text{for } i = \xi + 1, \ldots, n. \]  
(6.14)

**Proof.** Differentiating (4.12) with respect to \( t \) yields
\[ \frac{\partial \Psi_{\xi}}{\partial t} = \sum_{\rho=1}^{m} \left\{ \tilde{H}_\rho(x, y) \phi(x_\rho) + \left( \frac{\partial \phi(x_\rho)}{\partial \tilde{x}_\rho} \cdot \tilde{G}_\rho(x, y) \right) \tilde{y}_\rho \right\} \]
with a corresponding expression for \( \partial_i \Psi_u \). Hence (2.7b) holds if and only if
\[ \sum_{\rho=1}^{m} \left\{ \tilde{H}_\rho(x, y) \phi(x_\rho) + \left( \frac{\partial \phi(x_\rho)}{\partial \tilde{x}_\rho} \cdot \tilde{G}_\rho(x, y) \right) \tilde{y}_\rho \right\} \]
\[ = \sum_{\rho=1}^{n} \left\{ \tilde{H}_\rho(u, v) \phi(u_\rho) + \left( \frac{\partial \phi(u_\rho)}{\partial \tilde{u}_\rho} \cdot \tilde{G}_\rho(u, v) \right) \tilde{v}_\rho \right\} \]
or equivalently, recalling (6.9) and (6.10),
\[ 0 = \sum_{\rho=1}^{\xi} \left\{ (\tilde{H}_\rho(x, y) - \tilde{H}_\rho(u, v)) \phi(x_\rho) + \frac{\partial \phi(x_\rho)}{\partial \tilde{x}_\rho} \cdot (\tilde{G}_\rho(x, y) - \tilde{G}_\rho(u, v)) \tilde{y}_\rho \right\} \]
\[ + \sum_{\rho=\xi+1}^{m} \left\{ \tilde{H}_\rho(x, y) \phi(x_\rho) \right\} - \sum_{\rho=\xi+1}^{n} \left\{ \tilde{H}_\rho(u, v) \phi(u_\rho) \right\}. \]

By (2.8) and (2.9) these terms will not cancel, and therefore (2.7b) holds if and only if \( G \) and \( H \) satisfy (6.12), (6.13) and (6.14). \( \square \)

Note that there are no restrictions on \( \tilde{G}_i \) for \( i > \xi \) because these pods have zero weight and so do not contribute to \( \Psi_{\xi} \) or \( \Psi_u \).

The proof of theorem 2.3 can be simplified by the following result, which gives the form of \( G \) necessary for the existence of the flow-invariant subspaces of lemma 4.2.
Lemma 6.4. Let $F = (G, H) : X \times Y \to X \times Y$ be an $S_N$-equivariant mapping in $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$, where $S_N$ acts by (2.10). If $F$ has the property that for any isotropy subgroup $\Sigma \subseteq S_N$, the subspace $\text{Fix}(\Sigma|_X) \times Y$ is flow-invariant, then $G$ has the form

$$G(x, y) = \sum_{i=0}^\infty g_i([\alpha_i])^\infty_{\Sigma|_X=0} \hat{x}_i.$$  

(6.15)

That is, $j = 0$ in (6.5a).

Proof. Suppose that $x \in \text{Fix}(\Sigma|_X)$ and let $x_\eta$ and $x_\xi$ be two components such that $x_\eta = x_\xi$ for some $1 \leq \eta, \xi \leq N, \eta \neq \xi$. Then $x_\eta = x_\xi = \tilde{x}_t$ for some $1 \leq t \leq m$. Since $\text{Fix}(\Sigma|_X) \times Y$ is flow-invariant $G(x, y)$ must satisfy $G_\eta(x, y) = G_\xi(x, y)$. Hence, using the form for $G$ from (6.5a)

$$G_\eta(x, y) = \sum_{i, j=0}^\infty g_{ij}([\alpha_i])^\infty_{\Sigma|_X=0} \tilde{x}_i \tilde{y}_j = \sum_{i, j=0}^\infty g_{ij}([\alpha_i])^\infty_{\Sigma|_X=0} \tilde{x}_i \tilde{y}_j = G_\xi(x, y)$$

and therefore

$$\sum_{i, j=0}^\infty g_{ij}([\alpha_i])^\infty_{\Sigma|_X=0} \tilde{x}_i (\tilde{y}_j - \tilde{y}_j) = 0$$

for any $y_\eta, y_\xi \in R$. Since the $g_{ij}$ are arbitrary smooth functions, this is only true for all $y_\eta$ and $y_\xi$ if $j = 0$, giving (6.15). \qed

We are now in a position to prove theorem 2.3.

Proof of theorem 2.3. Let $\Sigma$ and $\Delta$ be the (possibly trivial) isotropy subgroups (6.8) and assume that $z = (x, y)$ and $w = (u, v)$ are such that $x \in \text{Fix}(\Sigma|_X),$ $u \in \text{Fix}(\Delta|_X).$ Suppose that $\Psi_\eta = \Psi_\eta$ (that is, (2.7a) is satisfied and (6.9) and (6.10) hold). We show that if $\delta_\psi \Psi_\eta = \delta_\psi \Psi_\eta$ (that is, if (2.7b) is satisfied) then $G$ and $H$ must have the forms in (2.12).

Note that if $\xi = 0$ in (6.9) then $\Psi_\eta = \Psi_\eta = 0$ and (2.7b) is satisfied trivially. So assume that $\xi \geq 1$ so that $\tilde{x}_\rho = \tilde{u}_\rho$ for all $\rho = 1, \ldots, \xi$. By (6.12), we require that $\tilde{G}_\rho(x, y) = \tilde{G}_\rho(u, v)$, and therefore, that $G_\eta(x, y) = G_\eta(u, v)$ for all $\eta$ and $\xi$ such that $\eta_\rho - 1 < \eta_\rho \leq \eta_\rho$ and $s_\rho - 1 < \xi \leq s_\rho$ (where $s_\rho = r_1 + \cdots + r_\rho$ and $s_0 = 1$ by analogy with the $q_\rho$ in (4.10)). Thus, using the form for $G_\xi$ given by (6.15), $G_\eta(x, y) = G_\xi(u, v)$ if and only if

$$\sum_{i=0}^\infty g_i([\alpha_i(x, y)])^\infty_{\Sigma|_X=0} \tilde{x}_i = \sum_{i=0}^\infty g_i([\alpha_i(u, v)])^\infty_{\Sigma|_X=0} \tilde{x}_i$$

and hence, since the $g_i$ are independent, if and only if

$$g_i([\alpha_i(x, y)])^\infty_{\Sigma|_X=0} = g_i([\alpha_i(u, v)])^\infty_{\Sigma|_X=0}$$

(6.16)

for all $i$. Thus, because the $g_i$ are arbitrary, $k$ and $l$ must be restricted so that (recalling the notation from (6.6))

$$\alpha_i(x, y) = \sum_{\rho=1}^k \tilde{x}_\rho \tilde{y}_\rho \tilde{y}_\rho + \sum_{\rho=1}^m \tilde{x}_\rho \tilde{v}_\rho = \sum_{\rho=1}^k \tilde{x}_\rho \tilde{y}_\rho + \sum_{\rho=1}^m \tilde{u}_\rho \tilde{v}_\rho = \alpha_i(u, v),$$

which holds if and only if

$$\sum_{\rho=1}^k \tilde{x}_\rho (\tilde{y}_\rho - \tilde{v}_\rho) + \sum_{\rho=1}^m \tilde{x}_\rho \tilde{v}_\rho - \sum_{\rho=1}^n \tilde{u}_\rho \tilde{v}_\rho = 0.$$
This must hold for all $\tilde{x}_\rho, \tilde{u}_\rho, \tilde{y}_\rho$ and $\tilde{v}_\rho$, so we require

\begin{align}
\tilde{x}_\rho^k (\tilde{y}_\rho^j - \tilde{v}_\rho^j) &= 0 \quad \text{for all } \rho = 1, \ldots, \xi \\
\tilde{x}_{\xi+1}^k \tilde{y}_{\xi+1}^j &= \cdots = \tilde{x}_m^k \tilde{y}_m^j = \tilde{u}_{\xi+1}^k \tilde{v}_{\xi+1}^j = \cdots = \tilde{u}_n^k \tilde{v}_n^j = 0. \quad (6.17)
\end{align}

Given (6.10), this is only true in general if $l = 1$. Therefore $G$ has the form given in (2.12a).

Turning now to (2.12b), consider first the implications of (6.14). Using the form for $H$ given in (6.5b), $\tilde{H}_\rho(x, y) = 0$ if and only if

\begin{align}
\sum_{s=q_{\rho, 1}}^{q_\rho} H_{\rho}(z) &= \sum_{i,j=0}^{\infty} h_{ij} \left( \{ \{ \alpha_{kl}(x) \} \}_{k,l=0}^{\infty} \right) \tilde{x}_{\rho}^i \tilde{y}_{\rho}^j = 0,
\end{align}

which requires that $\tilde{x}_{\rho}^i \tilde{y}_{\rho}^j = 0$ for all $\rho = \xi + 1, \ldots, m$. By (6.6) and (6.10), this is true in general only if $j = 1$.

Now consider the implications of (6.13). Setting $j = 1$ in (6.5b) and recalling (6.10), we find that (6.13) is satisfied for $\rho = 1, \ldots, \xi$ if and only if

\begin{align}
\sum_{i=0}^{\infty} h_{11} \left( \{ \{ \alpha_{kl}(x, y) \} \}_{k,l=0}^{\infty} \right) \tilde{x}_{\rho}^i \tilde{y}_{\rho} = \sum_{i=0}^{\infty} h_{11} \left( \{ \{ \alpha_{kl}(u, v) \} \}_{k,l=0}^{\infty} \right) \tilde{x}_{\rho}^i \tilde{y}_{\rho}.
\end{align}

That is

\begin{align}
\sum_{i=0}^{\infty} \left( h_{11} \left( \{ \{ \alpha_{kl}(x, y) \} \}_{k,l=0}^{\infty} \right) - h_{11} \left( \{ \{ \alpha_{kl}(u, v) \} \}_{k,l=0}^{\infty} \right) \right) \tilde{x}_{\rho}^i \tilde{y}_{\rho} = 0.
\end{align}

Since $\tilde{x}_{\rho}^i \tilde{y}_{\rho}$ is arbitrary, this requires

\begin{align}
h_{11} \left( \{ \{ \alpha_{kl}(x, y) \} \}_{k,l=0}^{\infty} \right) = h_{11} \left( \{ \{ \alpha_{kl}(u, v) \} \}_{k,l=0}^{\infty} \right)
\end{align}

for all $i = 1, \ldots, N - 1$. Thus, by the same argument from (6.16) to (6.17), the equality in (6.18) holds in general only if $l = 1$. Hence, $H$ must take the form in (2.12b).

\section{Conclusion and future directions}

In the preceding sections we have introduced, and to some extent explored, the pod systems of definition 2.1. The specified form of solutions in pod systems, given by the dynamic linear combination $\Psi_{\rho,1}(s)$ in (2.2), involves two components; the dynamic variables $z(t) = (x(t), y(t))$ and the pod function $\phi$. The dynamics of $z$ are given by the system of ODEs $\dot{z} = F(z)$ where $F$ is subject to the uniqueness condition (2.7). In order to make (2.7) meaningful for $\Psi_{\rho,1}$, we also imposed the linear independence conditions (2.8) and (2.9) on $\phi$. The main result of this paper, which underlines the usefulness of pod systems as a tool for analysis rather than numerical studies, is theorem 2.3, which gives an explicit pre-normal form for any $F$ that specifies the dynamics of a pod system.

Section 4 studied the consequences of (2.7) for $F$, which included equivalence under the action of $S_N$ in (2.10), for which we give a pre-normal form in theorem 4.1. An important additional property is the flow invariance of extended fixed-point subspaces (lemma 4.2), and we also demonstrated that dynamics in the extended fixed-point subspace of different isotropy subgroups are, in terms of $\Psi_{\rho,1}(s)$, equivalent. These observations formed the basis of the proof of theorem 2.3 in section 6.

In section 5, motivated in by the examples of section 3 and the discussion of section 4, we turned our attention to some of the features of solutions in pod systems. While pod systems have some difficulty in capturing spatially homogeneous solutions in a meaningful way, they...
are particularly well adapted to studying systems exhibiting multi-modal solutions if \( \phi \) is taken to be unimodal. In such cases, fully symmetric solutions, \( z \in \text{Fix}(S_N|_X) \times Y \), correspond to unimodal \( \Psi_1(s) \). In general, if \( \phi \) is not unimodal (and nowhere is it required to be) then fully symmetric solutions correspond to \( \Psi_1(s) = \tilde{\phi}(s; \tilde{x}) \) for some \( \tilde{y} \in R \), \( \tilde{x} \in R^d \). Thus pod systems offer a great degree of versatility in the choice of what constitutes a fully symmetric state, with \( \phi \) playing a role analogous to that of spatially homogeneous solutions in PDEs.

Thus pod functions complement PDEs as an approach to spatial dynamical systems, and as we have argued above, pod systems are in some ways more appropriate for macroscopic biological systems, just as PDEs are more appropriate for physical and chemical systems, largely because the low-energy states in the latter are considerably less important in the former. In the context of macroscopic biological systems, we are often concerned with the dynamics of functions in \( L^2 \), and proposition 5.1 showed that, in the limit of an infinite number of pods, \( \Psi_{s(w)}(s) \) can exhibit any state in \( L^1(R) \cap L^2(R) \cap C^\infty \).

Work on pod systems is currently proceeding with generic bifurcations from unimodal steady states in [8], with a view to building up a theory of spatial pattern formation that complements the existing PDE-based theory. Because of the symmetry component, bifurcations in pod systems bear some resemblance to the one-variable theory of [7], but there are substantial differences. The application of these results to the ecological and evolutionary theory of the formation of new species is pursued in [9], and we anticipate that pod systems will prove particularly useful for studying the dynamics of populations because populations are often clustered (in physical space or phenotype space).

Systems exhibiting travelling wave solutions may also be particularly susceptible to analysis in the pod framework. As a trivial example, if \( \phi \) specifies the constant shape of a wave, then the one-pod system \( \dot{x} = a, \dot{y} = (b - y)y \) has solutions \( \Psi_{s(w)}(s) = b\phi(s - (x_0 + at)) \) for initial conditions \( (x(0), y(0)) = (x_0, b) \).

Other directions for further study include building on section 4 to obtain a deeper understanding of the algebra of pod systems. Further understanding of the dynamics of pod systems—particularly behaviours such as in figure 3 for systems with infinite pods—is also desirable, and the analysis of some form of Fokker–Planck equations may well prove useful in this respect. Explicit extensions to more than one spatial dimension are also a pressing concern, and taking \( x_i \in R^d \) for \( d > 1 \) will indubitably be necessary for this; at the very least we would expect \( d \) to equal the spatial dimension of the system so that \( x_i \) may specify the location of the \( i \)th pod. Larger values of \( d \) are possible, but ensuring the linear independence conditions (2.8) and (2.9) then becomes more complex, particularly in the context of infinite pod systems.

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Appendix A. Proof of proposition 6.1

The proof of proposition 6.1 is similar to the proof of the corresponding result for \( S_N \)-equivariants in \( x = (x_1, \ldots, x_N) \) in [12, proposition 2.27]. Let \( p \) be any \( S_{N-1} \)-invariant
polynomial in \( x_2, \ldots, x_N \) and \( y_2, \ldots, y_N \) with degree \( k \). We claim that \( p \) can be written as

\[
p = \tilde{p} + \sum_{i=0}^{k} \sum_{i+j=0, i+j>0} D_{i,j} x_i^j y_j^j.
\]  

(A.1)

where \( \tilde{p} \) and the \( D_{i,j} \) are \( S_N \)-invariant polynomials in \( x_1, \ldots, x_N, y_1, \ldots, y_N \) and \( \tilde{p} \) has degree \( k \).

To prove the claim in (A.1), define

\[
\hat{\alpha}_{ij} = \sum_{k=2}^{N} x_k^i y_k^j
\]

for \( i, j = 0, \ldots, N-1 \) so that \( \hat{\alpha}_{ij} \) generate the \( S_{N-1} \)-invariants in, just as the \( \alpha_{ij} \) in (4.5) generate the \( S_N \)-invariants. By general theory for symmetric functions we can write \( p \) as

\[
p = G((\hat{\alpha}_{ij})_{i,j=0}^{N-1})
\]

and define

\[
\hat{p} = G((\alpha_{ij})_{i,j=0}^{N-1})
\]

so that

\[
\hat{p}(0, x_2, \ldots, x_N; 0, y_2, \ldots, y_N) = p(x_2, \ldots, x_N; y_2, \ldots, y_N).
\]

Clearly, \( \hat{p} \) has degree \( k \).

Although \( p - \hat{p} \) is not necessarily divisible by \( x_1 \) or \( y_1 \), we can expand \( p - \hat{p} \) as a polynomial in \( x_1 \)

\[
p - \hat{p} = A_0 + A_1 x_1 + \cdots + A_k x_1^k,
\]

where the \( A_i \) are polynomials in \( x_2, \ldots, x_N, y_1, \ldots, y_N \). The \( A_i \) can then be expanded as polynomials in \( y_1 \):

\[
A_i = B_{i,0} + B_{i,1} y_1 + \cdots + B_{i,r_i} y_1^{r_i},
\]

where the \( B_{ij} \) are \( S_{N-1} \)-invariant and \( B_{0,0} = 0 \) since \( p - \hat{p} \) cannot contain any terms which do not involve powers of \( x_1 \) or \( y_1 \). Hence

\[
p = \hat{p} + \sum_{i=0}^{k} \sum_{i+j=0, i+j>0} B_{i,j} x_i^j y_j^j,
\]  

(A.2)

where each \( B_{ij} \) has degree \( k_{ij} < k \) since the maximum degree of any term \( B_{ij} x_i^j y_j^j \) is \( k \) and \( B_{0,0} = 0 \).

Since the \( B_{ij} \) are \( S_{N-1} \)-invariant they can be expanded like \( p \) in (A.2) giving

\[
B_{ij} = \tilde{B}_{ij} + \sum_{l=0}^{k_i} \sum_{m=0}^{s_j} C_{lm} x_i^l y_j^m.
\]

Again, \( \tilde{B}_{ij} \) is \( S_N \)-invariant with the same degree as \( B_{ij} \), and the \( C_{lm} \) are \( S_{N-1} \)-invariant with lower degrees. Repeating the expansion until the degrees of the \( S_{N-1} \)-invariants are zero proves the claim in (A.1).
Now suppose that $P : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is $S_N$-equivariant with $P(x, y) = [Q(x, y)]$ where $Q(x, y)$ is $S_{N-1}$-invariant under permutations of $x_2, \ldots, x_N$ and $y_2, \ldots, y_N$. Then we can expand $Q(x, y)$ in $x_1$ and $y_1$ giving

$$Q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l_i} Q_{ij}x_1^i y_1^j,$$

where the $Q_{ij}$ are $S_{N-1}$-invariant. By the claim in (A.1) each $Q_{ij}$ can be written as

$$Q_{ij} = \tilde{Q}_{ij} + \sum_{m=0}^{r_i} \sum_{n=0}^{s_j} D_{mn} x_1^m y_1^n,$$

where $\tilde{Q}_{ij}$ and the $D_{mn}$ are $S_N$-invariant. Hence

$$Q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l_i} \left( \tilde{Q}_{ij} + \sum_{m=0}^{r_i} \sum_{n=0}^{s_j} D_{mn} x_1^m y_1^n \right) x_1^i y_1^j = \sum_{i=0}^{r_i} \sum_{j=0}^{s_j} R_{ij} x_1^i y_1^j,$$

where the $R_{ij}$ are $S_N$-invariant. Therefore

$$P(x, y) = [Q(x, y)] = \left[ \sum_{i=0}^{r_i} \sum_{j=0}^{s_j} R_{ij} x_1^i y_1^j \right] = \sum_{i=0}^{r_i} \sum_{j=0}^{s_j} R_{ij} \left[ x_1^i y_1^j \right] = \sum_{i=0}^{r_i} \sum_{j=0}^{s_j} R_{ij} E_{ij},$$

so that the $S_N$-equivariants are generated by the $E_{ij}$ over the $S_N$-invariants.

To see that only the $E_{ij}$, $0 \leq i, j \leq N - 1$, are necessary as generators, note that

$$(t - x_1)(t - x_2)\ldots(t - x_N) = t^N - \pi_1 t^{N-1} + \pi_2 t^{N-2} + \ldots + \pi_{N-1} t + \pi_N$$

where the $\pi_i$ are the elementary symmetric polynomials. Setting $t = x_1$ and $t = y_1$ in the corresponding formula for $y_1$, rearranging, and applying a standard inductive argument yields

$$x_1^m = R_0^m + R_1^m x_1 + \ldots + R_{N-2}^m x_1^{N-2} + R_{N-1}^m x_1^{N-1},$$

$$y_1^n = R_0^n + R_1^n y_1 + \ldots + R_{N-2}^n y_1^{N-2} + R_{N-1}^n y_1^{N-1},$$

for any $m, n > N$, where the $R_i^m$ and $R_j^n$ are $S_N$-invariant. Hence

$$x_1^m y_1^n = \left( \sum_{i=0}^{N-1} R_i^m x_1^i \right) \left( \sum_{j=0}^{N-1} R_j^n y_1^j \right) = \sum_{i,j=0}^{N-1} S_{ij} x_1^i y_1^j,$$

where the $S_{ij} = R_i^m R_j^n$ are $S_N$-invariants. □

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