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REDUCTIVE GROUP ACTIONS ON AFFINE QUADRICS WITH 1-DIMENSIONAL QUOTIENT: LINEARIZATION WHEN A LINEAR MODEL EXISTS

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Abstract. We study reductive group actions on complex affine quadrics. Such an action is called linearizable if it is equivalent to the restriction of a linear orthogonal action in the ambient affine space of the quadric. A linear model for a given action is a linear orthogonal action with the same orbit types and equivalent slice representations. We prove that if a reductive group action on an affine quadric with a 1-dimensional quotient has a linear model, then the action is linearizable. As a consequence, the action is linearizable if certain topological conditions are satisfied.

Introduction

1.1. Let G be a reductive complex algebraic group acting algebraically on a complex n-space $\mathbb{C}^n$. The linearization problem asks whether such actions are equivalent to a linear action, i.e., whether there is a G-equivariant isomorphism $\mathbb{C}^n \xrightarrow{\sim} V$, where V is a G-module ([Kr2]). Luna's slice theorem implies a positive answer if the quotient $\mathbb{C}^n/\!\!/G$ has dimension 0. In [Kr–S], Kraft and Schwarz studied the next complicated case of quotient dimension 1. This case is already highly nontrivial, and the first examples of nonlinearizable actions were found by Schwarz [S4] in this context. Using these results, Knop [Kn] proved that every noncommutative, connected reductive group has nonlinearizable actions on some $\mathbb{C}^n$.

In this paper we study an analogous problem for reductive group actions on the affine n-dimensional quadric $X := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = 1\} \subset \mathbb{C}^{n+1}$. If G is an algebraic group, every orthogonal representation $\rho : G \to \mathrm{O}_{n+1}(\mathbb{C})$ determines an action of G on X. These actions are called *linear actions,* and we say that an action of G on X is *linearizable* if it is conjugate within the group of algebraic automorphisms of X to a

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linear action of G on X.

Linearization Problem: Is every action of a reductive group on an affine quadric linearizable?

1.2. So far no example of a nonlinearizable reductive group action on X is known. However, we do not believe that every such action is linearizable, except under certain "smallness" assumptions. For example, every reductive group action on the 2-dimensional quadric as well as all actions with $\dim X/\!\!/G = 0$ are linearizable, see [D, 1.3 and 2.1].

In this paper we study actions with $\dim X/\!\!/G = 1$. Clearly, if linearization holds then there is a linear model in the following sense:

Definition. An orthogonal representation (V, G) is called a *linear model* for an action of G on X if X has the same orbit types and equivalent slice representations as the quadric $Q_V := \{v \in V \mid (v, v) = 1\} \subset V$ with the linear G-action.

The purpose of this paper is to prove that the converse is true for actions with a 1-dimensional quotient:

Theorem 1. *Let $\rho : G \times X \to X$ be an action of the connected reductive group G on the quadric X such that $\dim X/\!\!/G = 1$. Suppose that the action has a linear model. Then it is linearizable.*

1.3. In [D] we studied under which circumstances a linear model exists. We used a topological approach, and for the results, as well as for Theorem 1, we need only assume that X is an irreducible, smooth affine variety which is homotopy equivalent to a real sphere. We proved (see [D, 3.4]):

Proposition 1. *Let G act on X such that $\dim X/\!\!/G = 1$. Then we have:*

1. *$X/\!\!/G \cong \mathbb{A}$, the affine line.*
2. *There are two points $p, q \in \mathbb{A}$ such that the principal stratum is $\mathbb{A} \setminus \{p, q\}$.*
3. *The generic fiber of the quotient map (i.e., the fiber over the principal stratum) is a G-orbit, which means that the generic orbit is closed.*

Although we believe that the assumptions in Proposition 1 insure the existence of a linear model, we were able to prove existence only under the additional hypotheses (see [D, §4]):

Proposition 2. *Let G and X be as in above. Then a linear model exists in the following cases:*

1. *The G-action on X has a fixed point.*
2. *The principal isotropy group of the action is connected and the dimension of the slice representations is > 2.*

Together with Theorem 1 we obtain:

Theorem 2. *Let $\rho : G \times X \to X$ be an action of the connected reductive group G on the quadric X such that $\dim X/\!/G = 1$.*

1. *If ρ has a fixed point then ρ is linearizable.*
2. *If the principal isotropy group is connected and if the dimension of the slice representations of both exceptional isotropy groups is > 2, then ρ is linearizable.*

1.4. The proof of Theorem 1 follows the approach outlined in [Kr–S] by Kraft and Schwarz for the case of reductive group actions on the affine space $\mathbb{C}^n$. It consists of two steps. The first is to show that the fibrations over the principal stratum of the quotients corresponding to a given action and to its linear model are G-isomorphic. The second step is to show that there is, up to G-isomorphisms, only one way to glue the two exceptional fibers given by Proposition 1 into this fibration. For both steps knowledge of the automorphism group schemes that are associated with the linear models of quotient dimension 1 is crucial. The definition of these group schemes is recalled in Section 3. To determine their structure, one needs the classification of linear models, which was carried out in [D, §5]. The tables from [D] are repeated here in the Appendix.

Compared to the case of actions on affine space studied in [Kr–S], the present situation is more complicated due to the geometry of the quadric, which implies the existence of two exceptional orbits instead of just one. On the other hand, there are fewer linear models to be considered for the quadric, and in contrast to actions on affine space, it turns out that these linear actions already exhaust, up to equivalence, all actions with a 1-dimensional quotient which have a linear model.

1.5. To conclude the introduction, we state the conventions and notation valid throughout this paper. All the varieties will be defined over the complex numbers, and most of the time we will work with affine varieties. If W and Z are varieties over Y, $W(Z)$ denotes the set of morphisms $Z \to W$ over Y. Let G be a reductive algebraic group acting on an affine variety Z. We denote by $\mathcal{O}(Z)$ the $\mathbb{C}$-algebra of regular functions and by $\mathcal{O}(Z)^G$ the subalgebra of G-invariants. A famous theorem of Hilbert asserts that $\mathcal{O}(Z)^G$ is a finitely generated $\mathbb{C}$-algebra (see [Kr1, II.3.2]). Let $Z/\!/G$ denote the corresponding affine variety, and let $\pi_Z : Z \to Z/\!/G$ denote the morphism corresponding to the inclusion $\mathcal{O}(Z)^G \subset \mathcal{O}(Z)$.

Proposition (see [Kr1, II.3.2]).

1. *π_Z is surjective.*
2. *Every fiber of π_Z contains a unique closed G-orbit, hence π_Z sets up a bijection between the closed orbits in Z and the points of $Z/\!/G$.*

If V is an M-module where M is an algebraic group, we will also use the notation (V, M) to emphasize the group involved. Luna's slice theorem provides a strong link between general reductive group actions and representation theory. We will often use this important result. For a detailed treatment of the slice theorem we refer the reader to the original article [Lu] of Luna, or to the article [Sl] of Slodowy.

We make the following general assumptions. The connected reductive group G acts almost effectively on X such that $\dim X/\!\!/G = 1$, where X is a smooth affine variety which is homotopy equivalent to a real sphere. (Here almost effectively means that the kernel of the action is finite.) The action has a linear model (V, G), which is one of the representations in Tables 2–5 in the Appendix by [D, 5.1]. We denote by Q the quadric $Q_V := \{v \in V \mid (v, v) = 1\} \subset V$ with the linear G-action, and by $\ddot{\mathbb{A}} := \mathbb{A} \backslash \{p, q\}$ the principal stratum of the quotient $Q/\!\!/G = \mathbb{A}$, cf. Proposition 1 in 1.3. If u, v are the 2 exceptional points in $\mathbb{A} = X/\!\!/G$, there is a linear automorphism of $\mathbb{A}$ which sends u to p and v to q; hence by identifying $X/\!\!/G$ with $Q/\!\!/G$ we may assume that $u = p$ and $v = q$. We will need various local versions of the varieties X and Q. We denote by $\ddot{X} := \pi_X^{-1}(\ddot{\mathbb{A}})$ and $\ddot{Q} := \pi_Q^{-1}(\ddot{\mathbb{A}})$ the fibrations over the principal stratum with fibers isomorphic to the generic G-orbit (cf. Proposition 1 in 1.3). We define the variety $\hat{\mathbb{A}}_p$ by $\mathcal{O}(\hat{\mathbb{A}}_p) := \mathbb{C}[[z-p]]$, the formal power series in $z-p$. It is a formal complex analytic neighborhood of p in $\mathbb{A}$. We denote by $\hat{X}_p$ the pullback $X \times_{\mathbb{A}} \hat{\mathbb{A}}_p$, where $\hat{\mathbb{A}}_p \to \mathbb{A}$ is given by $\mathbb{C}[z-p] \hookrightarrow \mathbb{C}[[z-p]]$. $\hat{Q}_p$, $\hat{\mathbb{A}}_q$, $\hat{X}_q$ and $\hat{Q}_q$ are defined similarly. We define $\dot{\mathbb{A}}_p := \mathbb{A} \setminus \{p\}$ and $\dot{X}_p := X \times_{\mathbb{A}} \dot{\mathbb{A}}_p = X \setminus \pi_X^{-1}(p)$. $\dot{Q}_p$, $\dot{\mathbb{A}}_q$, $\dot{X}_q$ and $\dot{Q}_q$ are defined similarly. Finally, we define $\hat{\dot{\mathbb{A}}}_p$ as the schematic intersection $\hat{\mathbb{A}}_p \cap \dot{\mathbb{A}}_p$. Hence by $\mathcal{O}(\hat{\dot{\mathbb{A}}}_p) = \mathbb{C}((z-p))$, where $\mathbb{C}((z-p))$ denotes the Laurent series in $z-p$. $\hat{\dot{X}}_p$ is then given as the pullback $X \times_{\mathbb{A}} \hat{\dot{\mathbb{A}}}_p$, where $\hat{\dot{\mathbb{A}}}_p \to \mathbb{A}$ is induced by $\mathbb{C}[z-p] \hookrightarrow \mathbb{C}((z-p))$. $\hat{\dot{Q}}_p$, $\hat{\dot{\mathbb{A}}}_q$, $\hat{\dot{X}}_q$ and $\hat{\dot{Q}}_q$ are defined analogously.

Finally, we will use the following groups: A and B are the exceptional isotropy groups corresponding to the points p and q, and H is the principal isotropy group. We can assume that $H \subset A \cap B$. Let $N := N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G. Thus N is the group of G-isomorphisms of the generic orbit G/H. Similarly $N_A := N_A(H)/H$ and $N := N_B(H)/H$ are the A- resp. B-automorphism groups of the generic orbits in the slice representations of A and B. We often regard N_A and N_B as subgroups of N. Since the slice representations of A and B are orthogonal with the 1-dimensional quotient, Table 1 in the Appendix shows that N_A and N_B are isomorphic to either $\mathbb{C}^*$, SL_2 or $\mathbb{Z}_2$. (Applying the theorem of Luna–Richardson, see 3.4 below, to the slice representations shows that the reductive groups N_A and N_B are affine quadrics, which only leaves the three possibilities mentioned.)

2. A condition for linearizability

2.1. The strategy to attack the linearization problem is given by the next

Proposition. *X is G-isomorphic to Q if and only if the following holds:*

(1) *There are G-isomorphisms*

$$\ddot{\varphi} : \ddot{X} \xrightarrow{\sim} \ddot{Q},$$
$$\hat{\varphi}_p : \hat{X}_p \xrightarrow{\sim} \hat{Q}_p,$$
$$\hat{\varphi}_q : \hat{X}_q \xrightarrow{\sim} \hat{Q}_q$$

over $\ddot{\mathbb{A}}$, $\hat{\mathbb{A}}_p$ and $\hat{\mathbb{A}}_q$ respectively.

(2) *The isomorphisms $\ddot{\varphi}$, $\hat{\varphi}_p$ and $\hat{\varphi}_q$ induce the same isomorphisms*

$$\hat{\dot{\varphi}}_p := \ddot{\varphi}|_{\hat{\dot{X}}_p} = \hat{\varphi}_p|_{\hat{\dot{X}}_p} : \hat{\dot{X}}_p \xrightarrow{\sim} \hat{\dot{Q}}_p,$$
$$\hat{\dot{\varphi}}_q := \ddot{\varphi}|_{\hat{\dot{X}}_q} = \hat{\varphi}_q|_{\hat{\dot{X}}_q} : \hat{\dot{X}}_q \xrightarrow{\sim} \hat{\dot{Q}}_q.$$

Proof. Clearly, (1) and (2) hold if there is a G–isomorphism $X \to Q$. We show that the converse is also true. Taking intersections within $\mathcal{O}(\hat{\dot{\mathbb{A}}}_p)$ and $\mathcal{O}(\hat{\dot{\mathbb{A}}}_q)$ respectively, we have $\mathcal{O}(\ddot{\mathbb{A}}) \cap \mathcal{O}(\hat{\mathbb{A}}_p) = \mathcal{O}(\dot{\mathbb{A}}_q)$ and $\mathcal{O}(\ddot{\mathbb{A}}) \cap \mathcal{O}(\hat{\mathbb{A}}_q) = \mathcal{O}(\dot{\mathbb{A}}_p)$. Similarly, within $\mathcal{O}(\ddot{\mathbb{A}})$ we have $\mathcal{O}(\dot{\mathbb{A}}_p) \cap \mathcal{O}(\dot{\mathbb{A}}_q) = \mathcal{O}(\mathbb{A})$. Thus we get exact sequences:

$$0 \to \mathcal{O}(\mathbb{A}) \to \mathcal{O}(\dot{\mathbb{A}}_p) \oplus \mathcal{O}(\dot{\mathbb{A}}_q) \to \mathcal{O}(\ddot{\mathbb{A}}),$$
$$0 \to \mathcal{O}(\dot{\mathbb{A}}_p) \to \mathcal{O}(\ddot{\mathbb{A}}) \oplus \mathcal{O}(\hat{\mathbb{A}}_q) \to \mathcal{O}(\hat{\dot{\mathbb{A}}}_q),$$
$$0 \to \mathcal{O}(\dot{\mathbb{A}}_q) \to \mathcal{O}(\ddot{\mathbb{A}}) \oplus \mathcal{O}(\hat{\mathbb{A}}_p) \to \mathcal{O}(\hat{\dot{\mathbb{A}}}_p).$$

Since both quotients π_X and π_Q are given by a regular function, they are both flat. Tensoring the above sequences with $\mathcal{O}(X)$ and $\mathcal{O}(Q)$, we therefore get these pairs of exact sequences:

$$0 \to \mathcal{O}(X) \to \mathcal{O}(\dot{X}_p) \oplus \mathcal{O}(\dot{X}_q) \to \mathcal{O}(\ddot{X}),$$
$$0 \to \mathcal{O}(Q) \to \mathcal{O}(\dot{Q}_p) \oplus \mathcal{O}(\dot{Q}_q) \to \mathcal{O}(\ddot{Q}),$$

$$0 \to \mathcal{O}(\dot{X}_p) \to \mathcal{O}(\ddot{X}) \oplus \mathcal{O}(\hat{X}_q) \to \mathcal{O}(\hat{\dot{X}}_q),$$
$$0 \to \mathcal{O}(\dot{Q}_p) \to \mathcal{O}(\ddot{Q}) \oplus \mathcal{O}(\hat{Q}_q) \to \mathcal{O}(\hat{\dot{Q}}_q),$$

$$0 \to \mathcal{O}(\dot{X}_q) \to \mathcal{O}(\ddot{X}) \oplus \mathcal{O}(\hat{X}_p) \to \mathcal{O}(\hat{\dot{X}}_p),$$
$$0 \to \mathcal{O}(\dot{Q}_q) \to \mathcal{O}(\ddot{Q}) \oplus \mathcal{O}(\hat{Q}_p) \to \mathcal{O}(\hat{\dot{Q}}_p).$$

Assume that there is a G-isomorphism between the middle terms of such a pair of sequences, given by two G-isomorphisms between the direct summands. If the image of this isomorphism between the right hand terms of the pair of sequences is the identity, then the isomorphism induces a G-isomorphism between the left hand terms. Working our way upwards in the pairs of sequences above it then follows from assumption (2) that the G-isomorphisms $\ddot{\varphi}$, $\hat{\varphi}_p$, $\hat{\varphi}_q$ induce a global G-isomorphism $\varphi : X \to Q$. $\square$

2.2. To prove Theorem 1, we will find G-isomorphisms that satisfy the assumptions of 2.1. As a first step, the slice theorem gives G-isomorphisms $\hat{\psi}_p$ over $\hat{\mathbb{A}}_p$ and $\hat{\psi}_q$ over $\hat{\mathbb{A}}_q$ as in 2.1 (1). In fact, let (W, A) be the slice representation corresponding to the exceptional point p. Let $\hat{W}$ denote the (formal) completion of W along the fiber of the quotient $W \to \mathbb{A}$ that contains $0 \in W$. There is an induced action of A on $\hat{W}$, and we denote by $G \times^A \hat{W}$ the quotient of $G \times \hat{W}$ induced by this action and the A-action on G by right multiplication. Then the slice theorem implies that there are G-isomorphisms

$$\hat{Q}_p \xrightarrow{\sim} G \times^A \hat{W} \xleftarrow{\sim} \hat{X}_p.$$

It is easy to see that we can assume that these isomorphisms are maps over the quotient $\hat{\mathbb{A}}_p$. An analogous statement holds for the point q, whence the claim.

To proceed, we have to find G-isomorphisms $\ddot{\varphi}$ over the principal stratum, and then we have to glue the pieces together. For this we will study an affine group scheme, the automorphism group scheme $\mathcal{A}$ that is associated to the flat quotient $Q \to \mathbb{A}$. A G-isomorphism $\ddot{\varphi}$ is a section of the principal $\mathcal{A}$-bundle of fiberwise G-isomorphisms from $\ddot{X}$ to $\ddot{Q}$. Hence we have to show that this bundle is trivial. Fitting the local isomorphisms together to a global G-isomorphism translates to a condition on sections of the group scheme $\mathcal{A}$: multiplying $\hat{\varphi}_p$, $\hat{\varphi}_q$ and $\ddot{\varphi}$ by sections of $\mathcal{A}$ yields new G-isomorphisms, and we have to show that there are appropriate sections such that these new G-isomorphisms coincide locally and hence satisfy the second assumption in Proposition 2.1. This amounts to showing that there is, up to G-isomorphisms, a unique way to glue the two exceptional fibers over p and q into the principal fibration over $\ddot{\mathbb{A}}$.

3. The automorphism group scheme and linearization when its generic fiber is finite

3.1. Recall that an affine group scheme over Y is a variety $\mathcal{G}$ together with an affine morphism $\pi : \mathcal{G} \to Y$ and the structure of an algebraic group in each fiber of π such that this structure depends algebraically on $y \in Y$. For precise definitions see [D–G, III.4] or [Kr–S, III].

Let G be a reductive group and X an affine G-variety. Assume that the quotient map $\pi : X \to X/\!\!/G$ is flat. Given a morphism $\psi : Z \to X/\!\!/G$, we denote by X_Z the fiber product $X \times_{X/\!\!/G} Z$. Then G acts on X_Z, and the projection π_Z is the quotient map. Define the group

$$\operatorname{Aut}(X_Z)^G := \{\varphi : X_Z \xrightarrow{\sim} X_Z \mid \varphi \text{ is } G\text{−equivariant and } \pi_Z \circ \varphi = \pi_Z\}.$$

It is clear that this group depends functorially on Z. For the proof of the next proposition see [Kr–S, III.2.2].

Proposition. *The group functor* $Z \mapsto \operatorname{Aut}(X_Z)^G$ *is represented by an affine group scheme, the automorphism group scheme* $\pi : \mathcal{A}_X^G \to X /\!\!/ G$ *over* $X /\!\!/ G$*, i.e.,*

$$\operatorname{Aut}(X_Z)^G = \{\textit{morphisms } \sigma : Z \to \mathcal{A}_X^G \mid \pi \circ \sigma = \psi\}.$$

The group scheme $\mathcal{A}_X^G$ *is locally constant in the étale topology over every stratum of* $X /\!\!/ G$*.*

3.2. As an example, let (V, G) be an orthogonal representation with 1-dimensional quotient $V /\!\!/ G = \mathbb{A}$ (see Table 1 in the Appendix). This quotient is flat, and the principal stratum is $\dot{\mathbb{A}} = \mathbb{A} \setminus \{0\}$. The generic fiber of the automorphism group scheme $\mathcal{A}_V^G$, i.e., the fiber over a point of $\dot{\mathbb{A}}$ is isomorphic to the group $N = \operatorname{Aut}_G(G/H)$, where G/H is the generic fiber of the quotient. There is a faithful action of N on V commuting with G and stabilizing the invariant quadric, i.e., inducing G-automorphisms of V over the quotient, hence global sections of $\mathcal{A}_V^G$. It follows that the restriction $\mathcal{A}_V^G|_{\dot{\mathbb{A}}}$ to the principal stratum is constant. Moreover, if C denotes the closure of $\mathcal{A}_V^G|_{\dot{\mathbb{A}}}$ in $\mathcal{A}_V^G$, then $C \cong \mathbb{A} \times N$.

3.3. If Q is the quadric with a linear action and 1-dimensional quotient, the quotient map is flat, and the associated automorphism group scheme $\mathcal{A} := \mathcal{A}_Q^G$ is locally constant over the principal stratum $\ddot{\mathbb{A}}$ with fiber $N = \operatorname{Aut}_G(F)$, where $F \cong G/H$ is the principal fiber of the quotient map (cf. 1.5). We want to determine how the restriction of $\mathcal{A}$ to $\ddot{\mathbb{A}}$ is twisted around the two missing points.

Recall the definition of N_A and N_B as the normalizers of the principal isotropy group $H \subset A \cap B$ in the exceptional isotropy groups A and B, see 1.5. Let $\mathbb{B}$ be a copy of the affine line, and define $\dot{\mathbb{B}} := \mathbb{B} \setminus \{0\}$ and $\hat{\dot{\mathbb{B}}}$ by $\mathcal{O}(\hat{\dot{\mathbb{B}}}) := \mathbb{C}((z))$, the Laurent series in z.

Proposition. *There are subgroups* $\Gamma_A \subset N_A$ *and* $\Gamma_B \subset N_B$*, isomorphic to* $\mathbb{Z}_2$*, such that the following holds:*

$$\mathcal{A}|_{\hat{\dot{\mathbb{A}}}_p} \cong \hat{\dot{\mathbb{B}}} \times^{\Gamma_A} N,$$

$$\mathcal{A}|_{\hat{\dot{\mathbb{A}}}_q} \cong \hat{\dot{\mathbb{B}}} \times^{\Gamma_B} N,$$

where Γ_A *and* Γ_B *act on* N *by conjugation and on* $\hat{\dot{\mathbb{B}}}$ *by sending* z *to* $-z$*, and where* $\hat{\dot{\mathbb{B}}} \times^{\Gamma_A} N$ *and* $\hat{\dot{\mathbb{B}}} \times^{\Gamma_B} N$ *denote the quotients of* $\hat{\dot{\mathbb{B}}} \times N$ *by these actions.*

The actions of Γ_A *on* N_A *and* Γ_B *on* N_B *are trivial. Therefore, the subgroups* N_A *and* N_B *of* N*, viewed as sections of* $\mathcal{A}$ *over* $\hat{\dot{\mathbb{A}}}_p$ *and* $\hat{\dot{\mathbb{A}}}_q$*, extend over the points* p *and* q *respectively.*

Proof. Let (W, A) be the slice representation at p with the quotient map $\pi_W : W \to \mathbb{A}$, where we arrange that $\pi_W(0) = 0$. Let $F_W := \pi_W^{-1}(1) \cong A/H$ be the generic fiber of this map. Since π_W is given by a quadratic polynomial, there is an action of a subgroup $\Gamma_A \cong \mathbb{Z}_2$ of the group N_A of A-isomorphisms of F_W given by $-\,\mathrm{id}$. Γ_A also acts on $\mathbb{B}$ by $-\,\mathrm{id}$, and $\mathbb{A} = \mathbb{B}/\Gamma_A$. There is a canonical A-equivariant morphism

$$\begin{aligned} \rho : \mathbb{B} \times F_W &\to \mathbb{B} \times_{\mathbb{A}} W, \\ (z, w) &\mapsto (z, zw), \end{aligned}$$

over $\mathbb{B}$. If $\dot{\mathbb{B}} \times^{\Gamma_A} F_W$ denotes the quotient of $\dot{\mathbb{B}} \times F_W$ by Γ_A, it is easy to see that the map ρ induces an A-isomorphism

$$\begin{aligned} \dot{\rho} : \dot{\mathbb{B}} \times^{\Gamma_A} F_W &\to \dot{W}, \\ [z, w] &\mapsto zw, \end{aligned}$$

where $\dot{W} := W \setminus \{\pi_W^{-1}(0)\}$. From this we obtain a G-isomorphism, denoted by the same symbol:

$$\begin{aligned} \dot{\rho} : \dot{\mathbb{B}} \times^{\Gamma_A} (G \times^A F_W) &\to G \times^A \dot{W}, \\ [z, [g, w]] &\mapsto [g, zw]. \end{aligned}$$

Over $\hat{\dot{\mathbb{B}}}$ we get from this and the slice theorem a G-isomorphism

$$\hat{\dot{\rho}} : \hat{\dot{Q}}_p \to \hat{\dot{\mathbb{B}}} \times^{\Gamma_A} (G \times^A F_W)$$

over $\hat{\dot{\mathbb{B}}}$. Here $G \times^A F_W$ is isomorphic to the generic orbit F in Q. By functoriality of the group scheme $\mathcal{A}$ we obtain the first claim for the point p, and also for the point q in a completely analogous way.

The N_A-action on the fiber F_W extends to an action on W commuting with A by 3.2. Therefore, the Γ_A-action on $N_A \subset N$ must be trivial, and $N_A \subset \mathcal{A}(\hat{\mathbb{A}}_p)$, i.e., as sections of $\mathcal{A}$ over $\hat{\dot{\mathbb{A}}}_p$, N_A extends to p. Thus the subgroup $N_A \subset N$ is only twisted around the point q. An analogous statement holds for $N_B \subset N$. $\square$

3.4. In the rest of this section we prove linearization in case the generic fiber N of $\mathcal{A}$ is finite. Let (V, G) be the linear model, and let V^H denote the variety of fixed points of V under the action of the principal isotropy group H. By a theorem of Luna and Richardson (see [Lu–R, Thm. 4.2]) the quotients $V/\!\!/G$ and $V^H/\!\!/N$ are isomorphic. Thus V^H is 2-dimensional if N is finite. Because V^H has a unique N–stable complement in V, it follows that the restriction of Q to V^H is a nondegenerate N-invariant quadratic form, hence that $Q^H \cong \mathbb{C}^*$ and N is a finite subgroup of $\mathrm{O}_2(\mathbb{C}) = \mathbb{C}^* \rtimes \mathbb{Z}_2$. As such, it is either a cyclic subgroup of the connected component of the identity $\mathbb{C}^*$ of $\mathrm{O}_2(\mathbb{C})$, or a dihedral group $\mathcal{D}_{2n}$ for some $n \geq 1$, where $\mathcal{D}_{2n}$ is generated by the generator a of the subgroup $\mathbb{Z}_2$ in the decomposition of $\mathrm{O}_2(\mathbb{C})$ above, and by an element $b \in \mathbb{C}^*$ of order n, i.e., $\mathcal{D}_{2n} =< a, b \mid a^2 = b^n = 1,\ ab = b^{n-1}a >$. Since $Q^H/N = \mathbb{C}^*/N \cong Q/G = \mathbb{A}$, the former is impossible, hence N is a dihedral group $\mathcal{D}_{2n}$. Note that by considering the standard action of $\mathcal{D}_{2n}$ on $\mathbb{C}^2$ given by $a(x,y) = (y,x)$ and $b(x,y) = (bx, b^{-1}y)$, it is easy to conclude from the theorem of Luna and Richardson that if $f, g \in \mathcal{O}(V)^G$ are homogeneous generators of $\mathcal{O}(V)^G$ such that Q is the zero set of $f - 1$, then $n = \deg g$.

Proposition. *There is a G-isomorphism $\ddot{X} \xrightarrow{\sim} \ddot{Q}$ over $\ddot{\mathbb{A}}$.*

Proof. The bundles $\ddot{X}^H$ and $\ddot{Q}^H$ over $\ddot{\mathbb{A}}$ are the principal N-bundles associated with $\ddot{X}$ and $\ddot{Q}$. We have to show that there is an N-isomorphism $\ddot{X}^H \to \ddot{Q}^H$ over $\ddot{\mathbb{A}}$. We first show that the set of fixed points X^H is connected by applying Smith theory. The classification in the Appendix shows that, for every linear model with finite N, one of the following is true for H:

- H is a dihedral group;
- H is a torus;
- H contains a torus or an extension of a torus by $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ which has the same fixed points as H in the linear model Q, hence also in X.

(The only non obvious cases are the representations of C_3, F_4 and $C_2 \times C_m$ in Table 4, but by looking at the slice representations and using Luna's theorem, one sees that in these cases the third statement is true.) In each case we can apply Smith theory (see [B, Chap. III, §7 and §10]) to conclude that X^H has the $\mathbb{R}$-cohomology of a sphere, hence X^H is connected.

Applying the theorem of Luna and Richardson to X, we see that $X^H/N \cong \mathbb{A}$. Moreover, all but two of the fibers consist of $|N|$ points, and the two exceptional fibers consist of $\frac{|N|}{2}$ points. We therefore obtain for the Euler characteristic of X^H:

$$\chi(X^H) = n \cdot \chi(\mathrm{pt}) + n \cdot \chi(\mathrm{pt}) + 2n \cdot \chi(\ddot{\mathbb{A}}) = 0.$$

Thus X^H is an affine irreducible smooth curve with vanishing Euler characteristic, and it follows that $X^H \cong \mathbb{C}^*$. We already know that $Q^H \cong \mathbb{C}^*$. Since N is a dihedral group, there is a unique way to embed N into the automorphism group of $\mathbb{C}^*$ such that the quotient is isomorphic to the affine line $\mathbb{A}$, and it follows that there is an N-isomorphism $X^H \to Q^H$ over $\mathbb{A}$ whose restriction to $\ddot{\mathbb{A}}$ is an N-isomorphism $\ddot{X}^H \to \ddot{Q}^H$. ☐

3.5. Linearization now follows from the next

Proposition. *The following holds for sections of $\mathcal{A}$ over $\hat{\ddot{\mathbb{A}}}_p$ and $\hat{\ddot{\mathbb{A}}}_q$ respectively:*

$$\mathcal{A}(\hat{\ddot{\mathbb{A}}}_p) = \mathcal{A}(\hat{\mathbb{A}}_p),$$
$$\mathcal{A}(\hat{\ddot{\mathbb{A}}}_q) = \mathcal{A}(\hat{\mathbb{A}}_q).$$

Proof. By construction the principal isotropy group of N on Q^H is trivial, and there are two exceptional isotropy groups N_A and N_B, both isomorphic to $\mathbb{Z}_2$. Consider the action of N_i on $N = \mathcal{D}_{2n}$ via conjugation, $i = A, B$. One easily verifies the following: if n is odd, then $N^{N_i} = N_i$; if n is even, then the center $Z(N)$ of N is isomorphic to $\mathbb{Z}_2$, and $N^{N_i} = N_i \times Z(N)$. In the sequel, we will carry out the arguments for the point p, but everything will be the same for the point q. It is clear that the group Γ_A in Proposition 3.3 is just the group N_A in the present situation. Thus $\mathcal{A}(\hat{\ddot{\mathbb{A}}}_p) \cong \hat{\ddot{\mathbb{B}}} \times^{N_A} N$, where N_A acts on N by conjugation. By 3.3 we have $N_A \subset \mathcal{A}(\hat{\mathbb{A}}_p)$. If n is odd, we conclude that $\mathcal{A}(\hat{\ddot{\mathbb{A}}}_p) = N^{N_A} = N_A \subset \mathcal{A}(\hat{\mathbb{A}}_p)$, which implies the claim.

If n is even, then $\mathcal{A}(\hat{\ddot{\mathbb{A}}}_p) = N^{N_A} = N_A \times Z(N)$, and since $N_A \subset \mathcal{A}(\hat{\mathbb{A}}_p) \subset \mathcal{A}(\hat{\ddot{\mathbb{A}}}_p)$, we only have to show that there is a section in $\mathcal{A}(\hat{\mathbb{A}}_p)$ which is not in N_A. As mentioned in 3.4, the generating invariants of $\mathcal{O}(V)^G$ both have even degree if n is even. It follows that there is an action of $\mathbb{Z}_2$ on V, given by $\pm\,\mathrm{id}$, which commutes with G and fixes $\mathcal{O}(V)^G$, hence induces a global section of $\mathcal{A}$ of order 2. It also induces a section $w \in \mathcal{A}(\hat{\mathbb{A}}_p)$. It is clear that the section w, viewed as a G-automorphism of $\hat{Q}_p$ over $\hat{\mathbb{A}}_p$, has no fixed points, whereas it is equally clear that the automorphism of $\hat{Q}_p$ corresponding to the generator of N_A has fixed points, namely the closed orbit in the fiber over p. Thus, $w \notin N_A$ as a section over $\hat{\mathbb{A}}_p$. $\square$

Corollary. *Suppose the given G-action ρ on X has an orthogonal linear model such that $N = N_G(H)/H$ is finite. Then ρ is linearizable.*

Proof. Let $\hat{\psi}_p : \hat{Q}_p \to \hat{X}_p$, $\hat{\psi}_q : \hat{Q}_q \to \hat{X}_q$ and $\ddot{\psi} : \ddot{Q} \to \ddot{X}$ be the G-isomorphisms which exist by 2.3 and 3.4, and consider the sections $\hat{\psi}_p^{-1} \circ \ddot{\psi} \in \mathcal{A}(\hat{\ddot{\mathbb{A}}}_p)$ and $\hat{\psi}_q^{-1} \circ \ddot{\psi} \in \mathcal{A}(\hat{\ddot{\mathbb{A}}}_q)$. By the last proposition we have $\hat{\psi}_p^{-1} \circ \ddot{\psi} = \hat{\gamma}_p \in \mathcal{A}(\hat{\mathbb{A}}_p)$ and $\hat{\psi}_q^{-1} \circ \ddot{\psi} = \hat{\gamma}_q \in \mathcal{A}(\hat{\mathbb{A}}_q)$. Define new G-isomorphisms $\hat{\varphi}_p := \hat{\psi}_p \cdot \hat{\gamma}_p : \hat{Q}_p \to \hat{X}_p$ and $\hat{\varphi}_q := \hat{\psi}_q \cdot \hat{\gamma}_q : \hat{Q}_q \to \hat{X}_q$. Then the isomorphisms $\hat{\varphi}_p$, $\hat{\varphi}_q$ and $\ddot{\psi}$ satisfy both of the assumptions in Proposition 2.1, and the claim follows. $\square$

4. Linearization when N has positive dimension

4.1 We will now deal with the case that the group N is positive dimensional. A glance at the classification shows that this only happens if the linear model is reducible. We distinguish two subcases:

Type 1: The linear model is of the form $(W \oplus W^*, G)$, where W is a G-representation without invariants (Table 5 in the Appendix).

Type 2: The linear model is of the form $(V, G) = (V_1, G) \oplus (V_2, G)$, where the (V_i, G) are orthogonal representations with 1-dimensional quotient (Tables 2 and 3 in the Appendix).

4.2. The structure of the automorphism group schemes for linear models of Type 1 can be determined using the following remarks. Since the linear model (V, G) is orthogonal and has a 2-dimensional quotient, it follows from [S2] that the ring of regular functions $\mathcal{O}(V)$ is a graded free module over the ring of invariants $\mathcal{O}(V)^G$, and that there is a G-stable homogeneous subspace $T \subset \mathcal{O}(V)$ such that multiplication induces an isomorphism $\mathcal{O}(V)^G \otimes T \xrightarrow{\sim} \mathcal{O}(V)$. Let $V_1, \dots, V_r$ be the irreducible components of V. Then the sum T_i of all subspaces of T which are isomorphic to V_i^* is finite dimensional (see [S1]), and the multiplication induces an isomorphism $\mathcal{O}(V)^G \otimes T_i \xrightarrow{\sim} \mathcal{O}(V)^G \cdot T_i$ onto the isotypic component of $\mathcal{O}(V)$ of type V_i^*. The natural projection $\mathrm{pr} : \mathcal{O}(V) \to \mathcal{O}(Q)$ induces an isomorphism $\mathcal{O}(Q)^G \otimes S \xrightarrow{\sim} \mathcal{O}(Q)$, where $S = \mathrm{pr}(T)$ is isomorphic to T. If $S_i := \mathrm{pr}(T_i)$, it follows that the multiplication induces isomorphisms $\mathcal{O}(Q)^G \otimes S_i \xrightarrow{\sim} \mathcal{O}(Q)^G \cdot S_i$ onto the isotypic component of $\mathcal{O}(Q)$ of type V_i^*. It is shown in [Kr–S, III.2.3 and 2.4] that the group scheme $\mathcal{A}$ is completely determined by the sum of these isotypic components. The reason is that a G-automorphism over the quotient is determined by its action on the generating invariant and on the isotypic components of $\mathcal{O}(Q)$, i.e., by its action on the $\mathcal{O}(Q)^G$-module of all G-equivariant morphisms $V \to V$, or, equivalently, of the G-invariant vector fields $\Delta(V)^G$ on V. Consider now the representation (V^H, N) of N on the subspace of V that is fixed by the principal isotropy group H, where $N = N_G(H)/H$ as usual. Then the obvious restriction maps induce the following commutative diagram for invariant vector fields:

$$\begin{tikzcd}
\Delta(V)^G \arrow[r, "\mathrm{res}_V"] \arrow[d, "\mathrm{pr}"'] & \Delta(V^H)^N \arrow[d, "\mathrm{pr}"] \\
\Delta(Q)^G \arrow[r, "\mathrm{res}_Q"] & \Delta(Q^H)^N
\end{tikzcd}$$

The projections are surjective, and res_V is an isomorphism if and only if res_Q is an isomorphism. It follows from the above that if res_V is an isomorphism, then the automorphism group schemes of Q and of Q^H are isomorphic.

4.3. For all linear models of Type 1 res_V is an isomorphism. For the first entry of Table 5 this is straightforward; for the remaining three entries it is contained in the results of §14 in [S3] (see Table I, entries 4 and 7′, and Table IV, entry 19 in [S3]). Thus, for linear models of Type 1 the automorphism group schemes are those associated with the N-action on the invariant quadric $Q^H \subset V^H$ in the representations (V^H, N). For all Type 1 models, these representations are equivalent to the following: $V^H = \mathbb{C}^4$, $N = \mathbb{C}^* \times (\mathbb{C}^* \rtimes \mathbb{Z}_2)$, and the action is given by $(t,s)(x,y,z,w) = (tsx, ts^{-1}y, t^{-1}sz, t^{-1}s^{-1}w)$ for $(t,s) \in \mathbb{C}^* \times \mathbb{C}^*$ and by $\gamma(x,y,z,w) = (y,x,w,z)$ for the generator γ of $\mathbb{Z}_2$. The quadric Q^H is given by $Q^H = \{(x,y,z,w) \in V^H \mid xw + yz = 1\}$. Note first of all that there is a $\mathbb{C}^*$-action on V^H that commutes with N and fixes the invariants. It is given by the connected component of the identity in the center of N. It follows that the automorphism group scheme $\mathcal{A}$ contains a central subgroup $\mathcal{M} \cong \mathbb{A} \times \mathbb{C}^*$. For the isotropy subgroups A and B corresponding to the cexceptional points p and q, we have $A = N_A \cong \mathbb{C}^*$ and $B = N_B \cong \mathbb{Z}_2$ (w.l.o.g.), and $N_A \rtimes N_B$ corresponds to the factor $\mathbb{C}^* \rtimes \mathbb{Z}_2$ of N.

Let N^0 denote the connected component of the identity of N. Then $N^0 \cong \mathbb{C}^* \times \mathbb{C}^*$ and $N/N^0 \cong \mathbb{Z}_2$. Consider the restriction (V^H, N^0) of (V^H, N) to N^0 and the corresponding linear action of N^0 on Q^H. Again we have $Q^H /\!\!/ N^0 \cong \mathbb{B}$, where $\mathbb{B}$ denotes the affine line. The two exceptional points p' and q' of this quotient correspond to isotropy groups isomorphic to $\mathbb{C}^*$. Let $\mathcal{C}$ denote the automorphism group scheme of the N^0-action on Q^H. This group scheme is essentially constant: the closure of the restriction $\mathcal{C}|_{\ddot{\mathbb{A}}}$ (where $\ddot{\mathbb{A}} = \mathbb{A} \setminus \{p', q'\}$) is isomorphic to $\mathbb{A} \times N^0$. The group N/N^0 acts on $\mathbb{B}$ by identifying p' and q', the corresponding point of the quotient $\mathbb{B}/(N/N^0) = \mathbb{A} = Q^H /\!\!/ N$ being the exceptional point p. The exceptional point q of this quotient is the image of the unique fixed point of the N/N^0-action on $\mathbb{B}$. If $\mathcal{A}^0$ denotes the connected component of the identity section of the group scheme $\mathcal{A}$ that corresponds to the N-action on V^H, it follows that

$$\mathcal{A}^0 \cong \mathbb{B} \times^{N/N^0} \mathcal{C}.$$

N/N^0 acts trivially on the center of N, and nontrivially on the other factor $\mathbb{C}^*$ of $\mathcal{C}$ that corresponds to N_A. It follows that the restriction $\mathcal{A}^0|_{\dot{\mathbb{A}}_q}$, where $\dot{\mathbb{A}}_q = \mathbb{A} \setminus \{q\}$ contains a normal subgroup scheme $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\Gamma_B} N_A$, where $N/N^0 = \Gamma_B = N_B \cong \mathbb{Z}_2$ (cf. Proposition 3.3). Moreover, since the subgroup N_B extends as sections of $\mathcal{A}$ over the point q by Proposition 3.3, it follows that

$$\mathcal{A}/\mathcal{A}^0|_{\dot{\mathbb{A}}_p} \cong \dot{\mathbb{A}}_p \times \mathbb{Z}_2,$$

where $\dot{\mathbb{A}}_p = \mathbb{A} \setminus \{p\}$. These remarks prove part (1) of the next proposition.

Proposition.

(1) *For linear models of Type* 1, *let* $\mathcal{A}^0$ *denote the connected component of the identity section of* $\mathcal{A}$. *Then* $\mathcal{A}/\mathcal{A}^0|_{\dot{\mathbb{A}}_p} \cong \dot{\mathbb{A}}_p \times \mathbb{Z}_2$. *Let* $\dot{\mathbb{B}}$, Γ_B *and* N_A *be as in Proposition* 3.3. *Over* $\dot{\mathbb{A}}_q$, $\mathcal{A}^0$ *contains a subgroup scheme* $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\Gamma_B} N_A$. *Moreover,* $\mathcal{A}$ *contains a central subgroup* $\mathcal{M} \cong \mathbb{A} \times \mathbb{C}^*$, *and there is an exact sequence of group schemes over* $\dot{\mathbb{A}}_q$

$$1 \to \dot{\mathcal{A}}_q \to \mathcal{A}^0|_{\dot{\mathbb{A}}_q} \to \mathcal{M}|_{\dot{\mathbb{A}}_q} \to 1.$$

(2) *For linear models of Type* 2, *let* $\dot{\mathbb{B}}$, Γ_A, Γ_B, N_A *and* N_B *be as in Proposition* 3.3. *Then the following holds:*
  - *Over* $\dot{\mathbb{A}}_p$, $\mathcal{A}$ *contains a subgroup* $\dot{\mathcal{A}}_p \cong \dot{\mathbb{A}}_p \times N_B$.
  - *Over* $\dot{\mathbb{A}}_q$, $\mathcal{A}$ *contains a subgroup* $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\Gamma_B} N_A$.
  - *Over* $\ddot{\mathbb{A}}$, *there is an exact sequence*

$$1 \to \dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}} \to \mathcal{A}|_{\ddot{\mathbb{A}}} \to (\mathcal{A}/\dot{\mathcal{A}}_q)|_{\ddot{\mathbb{A}}} \cong \dot{\mathcal{A}}_p|_{\ddot{\mathbb{A}}} \to 1.$$

(*Of course, these statements hold up to interchanging* p *and* q.)

Proof. We are left with proving (2). Let $(V,G) = (V_1,G) \oplus (V_2,G)$ be a linear model of Type 2, where (V_i,G), $i = 1,2$, are orthogonal representations with 1-dimensional quotients. Then the quadric $Q \subset V$ is given as the set of points $v \in V$ such that $f(v) + g(v) = 1$, where f and g are the quadratic G-invariants on V_1 and V_2. We may assume that the quotient $Q \to \mathbb{A}$ is given by the function g. Then the exceptional points of the quotient are $p = 1$ and $q = 0$. The exceptional isotropy group A corresponding to p is the principal isotropy group of the G-action on V_2, and its slice representation is (V_1, A). Similarly, the exceptional isotropy group B corresponding to the point q is the principal isotropy group of the G-action on V_1, and its slice representation is (V_2, B). Let $c \neq 0, 1$, let $F_1 = f^{-1}(1-c)$ be a generic G-orbit in V_1, and let $F_2 = g^{-1}(c)$ be a generic G-orbit in V_2. Then, clearly, $F_1 \times F_2$ is a generic G-orbit in Q.

Let $\mathbb{D}$ be a copy of the affine line. As in the proof of Proposition 3.3, we get a G-isomorphism

$$\begin{aligned} \dot{\mathbb{D}} \times^{\Gamma_A} F_1 &\to \dot{V}_1, \\ [t, v_1] &\mapsto t \cdot v_1, \end{aligned}$$

where $V_1 := V_1 \setminus \{\text{nullcone}\}$, and where Γ_A is a subgroup of $N_A = N_A(H)/H$ which is isomorphic to $\mathbb{Z}_2$ and whose generator acts on V_1 as $-\operatorname{id}$. Similarly, we get a G-isomorphism

$$\begin{aligned} \dot{\mathbb{B}} \times^{\Gamma_B} F_2 &\to \dot{V}_2, \\ [z, v_2] &\mapsto z \cdot v_2, \end{aligned}$$

where $\dot{\mathbb{B}} \cong \dot{\mathbb{D}}$. Consider the maps $\dot{\mathbb{D}} \to \dot{\mathbb{A}}$ given by $t \mapsto -t^2(1-c) + 1$ and $\dot{\mathbb{B}} \to \dot{\mathbb{A}}$ given by $z \mapsto cz^2$ and let $Y := \dot{\mathbb{D}} \times_{\ddot{\mathbb{A}}} \dot{\mathbb{B}}$ be the pullback over $\ddot{\mathbb{A}}$ with $Y \to \ddot{\mathbb{A}}$ defined by $(z,t) \mapsto cz^2$, where $t \neq 0, \frac{1}{\sqrt{1-c}}, \frac{-1}{\sqrt{1-c}}$ and $z \neq 0, \frac{1}{\sqrt{c}}, \frac{-1}{\sqrt{c}}$. Then the two G-isomorphisms from above induce a G-isomorphism

$$\begin{aligned} \psi : Y \times^{\Gamma_A \times \Gamma_B} (F_1 \times F_2) &\to \ddot{Q}, \\ [(t,z), (v_1, v_2)] &\mapsto (tv_1, zv_2), \end{aligned}$$

over $\ddot{\mathbb{A}}$ (where $\ddot{Q} \to \ddot{\mathbb{A}}$ is given by g and $Y \times^{\Gamma_A \times \Gamma_B} (F_1 \times F_2)$ is induced by $Y \to \ddot{\mathbb{A}}$). By functoriality, we get an isomorphism over $\ddot{\mathbb{A}}$

$$\varphi : Y \times^{\Gamma_A \times \Gamma_B} N \to \mathcal{A}|_{\ddot{\mathbb{A}}},$$

where N is the group of G-isomorphisms of the generic fiber $F_1 \times F_2$ over $Q/\!\!/G$. The actions of Γ_A and Γ_B on N are induced by their actions by $-\mathrm{id}$ on F_1 respectively F_2. The induced action of Γ_A is trivial on $N_A \subset N$, and it follows that $Y \times^{\Gamma_A \times \Gamma_B} N$ contains a subgroup $Y \times^{\Gamma_B} N_A$. Since Γ_B does not act on $\dot{\mathbb{D}}$, it is clear that this subgroup extends over the point p to a group scheme $\dot{\mathbb{B}} \times^{\Gamma_B} N_A$ over $\dot{\mathbb{A}}_q$. On the other hand, we know that the group N_A also extends over the point p as sections in $\mathcal{A}(\hat{\dot{\mathbb{A}}}_p)$, see 3.3. It follows that the isomorphism φ extends to an isomorphism of $\dot{\mathbb{B}} \times^{\Gamma_B} N_A$ onto a subgroup scheme $\dot{\mathcal{A}}_q \subset \mathcal{A}|_{\dot{\mathbb{A}}_q}$. Similarly, over the point q, φ extends to an isomorphism of $\dot{\mathbb{D}} \times^{\Gamma_A} N_B$ onto a subgroup scheme $\dot{\mathcal{A}}_p \subset \mathcal{A}|_{\dot{\mathbb{A}}_p}$.

For linear models of Type 2 it follows from the classification or with the help of the theorem of Luna and Richardson that $N = N_A \cdot N_B$, i.e., that every $n \in N$ can be written as $n = n_A \cdot n_B$ with $n_A \in N_A$ and $n_B \in N_B$. The classification also shows that we can assume that N_A is a positive dimensional connected normal subgroup of N. In fact, the connected component N^0 of the identity of N is a group of rank ≤ 2 that is not simple, hence one of the following: $\mathbb{C}^*$, SL_2, $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{C}^* \times \mathrm{SL}_2$, $\mathrm{SL}_2 \times \mathrm{SL}_2$. It is now easy to check that we can assume that N_A is one of the factors of N^0. If N^0 has two factors then $N^0 = N$, and N_A is normal in N. If N^0 has only one factor then $N_A = N^0$ is again normal in N. The exact sequence in claim (2) now follows from $N = N_A \cdot N_B$ if we can show that $\dot{\mathcal{A}}_p$ is constant. Recall that Γ_A lies in the center of N_A. If N is connected then, by the above, the center of N_A lies in the center of N, and if N is not connected, then $N = N_A \rtimes N_B$ with $N_B \cong \mathbb{Z}_2$. In both cases $\Gamma_A \subset N_A$ acts trivially on N_B, whence the claim. $\square$

4.4. Consider the functor which associates to each morphism $Y \to \ddot{\mathbb{A}}$ the set of G-isomorphisms $Y \times_{\ddot{\mathbb{A}}} \ddot{X} \to Y \times_{\ddot{\mathbb{A}}} \ddot{Q}$ over $\ddot{\mathbb{A}}$. It is represented by a variety $\mathcal{P} \to \ddot{\mathbb{A}}$ (see [D–G, III, §4]). If $\ddot{\mathcal{A}}$ denotes the restriction $\mathcal{A}|_{\ddot{\mathbb{A}}}$, then $\ddot{\mathcal{A}}$ acts on $\mathcal{P}$ from the right. The slice theorem implies that there is a variety Z and an étale surjective morphism $Z \to \ddot{\mathbb{A}}$ such that $Z \times_{\ddot{\mathbb{A}}} \mathcal{P}$ and $Z \times_{\ddot{\mathbb{A}}} \ddot{\mathcal{A}}$ are isomorphic as $Z \times_{\ddot{\mathbb{A}}} \ddot{\mathcal{A}}$-varieties with the obvious actions. That is, $\mathcal{P}$ is a principal $\ddot{\mathcal{A}}$-bundle, i.e., an element of the cohomology set $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \ddot{\mathcal{A}})$ of principal $\ddot{\mathcal{A}}$-bundles over $\ddot{\mathbb{A}}$ (cf. [D–G, III, §4]). A G-isomorphism $\ddot{X} \to \ddot{Q}$ is the same as a section $\ddot{\mathbb{A}} \to \mathcal{P}$, and the existence of such a section is equivalent to $\mathcal{P}$ being the trivial bundle, i.e., isomorphic to $\ddot{\mathcal{A}}$.

Proposition. *$\mathcal{P}$ is trivial, hence there is a G-isomorphism $\ddot{X} \to \ddot{Q}$ over $\ddot{\mathbb{A}}$.*

Proof. Consider first linear models of Type 1. There is an exact sequence

$$H^1_{\text{ét}}(\ddot{\mathbb{A}}, \mathcal{A}^0|_{\ddot{\mathbb{A}}}) \to H^1_{\text{ét}}(\ddot{\mathbb{A}}, \ddot{\mathcal{A}}) \xrightarrow{\beta} H^1_{\text{ét}}(\ddot{\mathbb{A}}, (\mathcal{A}/\mathcal{A}^0)|_{\ddot{\mathbb{A}}}).$$

The image $\beta(\mathcal{P})$ is a $\mathbb{Z}_2$-bundle over $\ddot{\mathbb{A}}$ by Proposition 4.3 (1). This bundle has sections around the two missing points in $\ddot{\mathbb{A}}$ because of the slice theorem. Therefore it is trivial, hence $\mathcal{P}$ is induced by an element $\mathcal{P}'$ of $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \mathcal{A}^0)$. The exact sequence in (4.3)(1) implies an exact sequence in cohomology:

$$H^1_{\text{ét}}(\ddot{\mathbb{A}}, \dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}}) \longrightarrow H^1_{\text{ét}}(\ddot{\mathbb{A}}, \mathcal{A}^0|_{\ddot{\mathbb{A}}}) \xrightarrow{\beta'} H^1_{\text{ét}}(\ddot{\mathbb{A}}, \mathcal{M}|_{\ddot{\mathbb{A}}}).$$

Now $\mathcal{M}|_{\ddot{\mathbb{A}}} \cong \ddot{\mathbb{A}} \times \mathbb{C}^*$. Hence the image $\beta'(\mathcal{P}')$ is a $\mathbb{C}^*$-bundle over $\ddot{\mathbb{A}}$ and therefore trivial, because being a smooth affine rational curve $\ddot{\mathbb{A}}$ has a trivial Picard group. Thus $\mathcal{P}'$ is induced by an element $\mathcal{Q}$ of $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}})$. We want to show that $\mathcal{Q}$ is the trivial bundle. Recall that $\dot{\mathcal{A}}_q \cong \mathbb{B} \times^{\mathbb{Z}_2} N_A$. Assuming, as we may, that $p = 1$ and $q = 0$, it follows that $\dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}}$ becomes constant over a double cover of the form $\tilde{\mathbb{B}} \to \ddot{\mathbb{A}}$, where $\tilde{\mathbb{B}} := \mathbb{B} \setminus \{0, 1, -1\}$, and where the covering map is given by $z \mapsto z^2$: $\tilde{\mathbb{B}} \times_{\ddot{\mathbb{A}}} \dot{\mathcal{A}}_q \cong \tilde{\mathbb{B}} \times \mathbb{C}^*$. But then the pullback $\tilde{\mathbb{B}} \times_{\ddot{\mathbb{A}}} \mathcal{Q}$ is again a $\mathbb{C}^*$-bundle over the smooth affine rational curve $\tilde{\mathbb{B}}$ and hence trivial. Thus $\mathcal{Q}$ belongs to the set of principal bundles of $\dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}}$ which become trivial when lifted to $\tilde{\mathbb{B}}$. It is known that the elements of this set correspond bijectively to the elements of the 1-cohomology set of the covering group $\mathbb{Z}_2$ with values in the space of sections $\tilde{\mathbb{B}} \to \tilde{\mathbb{B}} \times_{\ddot{\mathbb{A}}} \dot{\mathcal{A}}_q$ (see e.g. [Kr–S, IV.4.3]). Since $\tilde{\mathbb{B}} \times_{\ddot{\mathbb{A}}} \dot{\mathcal{A}}_q \cong \tilde{\mathbb{B}} \times \mathbb{C}^*$, this latter space consists of units in $\mathcal{O}(\tilde{\mathbb{B}})$. Thus it is enough to show that $H^1(\mathbb{Z}_2, \mathcal{O}(\tilde{\mathbb{B}})^*)$ is trivial. Now $\mathcal{O}(\tilde{\mathbb{B}})^* = \mathbb{C}[z, z^{-1}, (z-1)^{-1}, (z+1)^{-1}]^*$. The generator γ of $\mathbb{Z}_2$ acts on an element $h(z) = \lambda \cdot z^a (z-1)^b (z+1)^c \in \mathcal{O}(\tilde{\mathbb{B}})^*$ (where $\lambda \in \mathbb{C}^*$, $a, b, c \in \mathbb{Z}$) by

$${}^{\gamma}h(z) = h(-z)^{-1} = \lambda^{-1} z^{-a} (z-1)^{-c} (z+1)^{-b}.$$

To calculate $H^1(\mathbb{Z}_2, \mathcal{O}(\tilde{\mathbb{B}})^*)$, one has to consider the elements $D = \gamma - 1$ and $N = \gamma + 1$ of the group algebra over $\mathbb{Z}_2$. These elements operate on $\mathcal{O}(\tilde{\mathbb{B}})^*$ by

$$D(h(z)) = {}^{\gamma}h(z) \cdot h(z)^{-1},$$
$$N(h(z)) = {}^{\gamma}h(z) \cdot h(z).$$

By [Se, Chap.VIII, §4], we then have

$$H^1(\mathbb{Z}_2, \mathcal{O}(\tilde{\mathbb{B}})^*) = \operatorname{Ker}(N)/\operatorname{Im}(D).$$

An easy calculation shows that indeed $\operatorname{Im}(D) = \operatorname{Ker}(N)$, whence the claim.

For linear models of Type 2, it follows from the exact sequence in Proposition 4.3 (2) that it is enough to show that $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \dot{\mathcal{A}}_p|_{\ddot{\mathbb{A}}})$ and $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}})$ contain only the trivial element. Since $\dot{\mathcal{A}}_p$ is constant, the first claim follows again because $\ddot{\mathbb{A}}$ is a smooth affine rational curve. If $N_A \cong \mathbb{C}^*$, the second claim follows as for linear models of Type 1. Else we have $N_A \cong \mathrm{SL}_2$.

An easy exercise shows that the following is true: suppose the cyclic group $\Gamma \subset \dot{\mathbb{B}}$ acts on $\dot{\mathbb{B}}$ in the natural way and on the algebraic group L by group automorphisms. Then the group scheme $\dot{\mathbb{B}} \times^{\Gamma} L$ is constant if the action of Γ on L extends to an action of $\dot{\mathbb{B}}$. In particular, this is the case if the automorphism group of L is connected (embed Γ in a maximal torus of $\mathrm{Aut}(L)$). In the present situation it follows that $\dot{\mathcal{A}}_q$ is constant if $N_A \cong \mathrm{SL}_2$ because $\mathrm{Aut}(\mathrm{SL}_2) = \mathrm{PSL}_2$. But then $H^1_{\text{ét}}(\ddot{\mathbb{A}}, \dot{\mathcal{A}}_q|_{\ddot{\mathbb{A}}})$ is trivial, again because $\ddot{\mathbb{A}}$ is a smooth affine rational curve. $\square$

4.5. By 2.2 and by the last proposition, for all linear models of Type 1 and 2 we now have G-isomorphisms $\ddot{\psi} : \ddot{Q} \to \ddot{X}$ over $\ddot{\mathbb{A}}$, $\hat{\psi}_p : \hat{Q}_p \to \hat{X}_p$ over $\hat{\mathbb{A}}_p$ and $\hat{\psi}_q : \hat{Q}_q \to \hat{X}_q$ over $\hat{\mathbb{A}}_q$. By Proposition 2.1, to get linearization we have to multiply these isomorphisms by appropriate sections of the group scheme $\mathcal{A}$ so that $\ddot{\psi}^{-1} \circ \hat{\psi}_p \in \mathcal{A}(\hat{\dot{\mathbb{A}}}_p)$ and $\ddot{\psi}^{-1} \circ \hat{\psi}_q \in \mathcal{A}(\hat{\dot{\mathbb{A}}}_q)$ reduce to the identity section. First of all we remark that it is enough to consider the connected component $\mathcal{A}^0$ of $\mathcal{A}$. For if $\mathcal{A}$ is not connected, then by 4.3, $(\mathcal{A}/\mathcal{A}^0)|_{\ddot{\mathbb{A}}} \cong \ddot{\mathbb{A}} \times \mathbb{Z}_2$, and the section corresponding to the generator of $\mathbb{Z}_2$ extends over one of the exceptional points. By multiplying $\ddot{\psi}$ with this section, and, if necessary, also the G-isomorphism given in a neighborhood of the exceptional point to which the section extends, we can always arrange that $\ddot{\psi}^{-1} \circ \hat{\psi}_p$ and $\ddot{\psi}^{-1} \circ \hat{\psi}_q$ are sections of $\mathcal{A}^0$.

The proposition below gives a criterion when the glueing is possible. It disentangles the problem and reduces it to separate questions about the group schemes $\dot{\mathcal{A}}_q$ and $\dot{\mathcal{A}}_p$ (resp. $\mathcal{M}$) occurring in the exact sequences of Proposition 4.3. For linear models of both types let $\mathcal{B}_q$ denote the closure of $\dot{\mathcal{A}}_q$ in $\mathcal{A}$ (recall that $\dot{\mathcal{A}}_q$ was defined over $\dot{\mathbb{A}}_q = \mathbb{A} \setminus \{q\}$). For linear models of Type 2 let $\mathcal{B}_p$ denote the closure of $\dot{\mathcal{A}}_p$ in $\mathcal{A}$, and for models of Type 1 let $\mathcal{B}_p$ denote $\mathcal{M}$ (see Proposition 4.3). Consider the following conditions $(*)$:

$$\mathcal{B}_q(\hat{\dot{\mathbb{A}}}_p) \subset \mathcal{B}_q(\ddot{\mathbb{A}}) \cdot \mathcal{B}_q(\hat{\mathbb{A}}_p), \tag{$*1$}$$

$$\mathcal{B}_p(\hat{\dot{\mathbb{A}}}_q) \subset \mathcal{B}_p(\ddot{\mathbb{A}}) \cdot \mathcal{B}_p(\hat{\mathbb{A}}_q), \tag{$*2$}$$

$$\mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q) \subset \mathcal{B}_q(\dot{\mathbb{A}}_q) \cdot \mathcal{B}_q(\hat{\mathbb{A}}_q), \tag{$*3$}$$

$$\mathcal{B}_p(\hat{\dot{\mathbb{A}}}_p) \subset \mathcal{B}_p(\dot{\mathbb{A}}_p) \cdot \mathcal{B}_p(\hat{\mathbb{A}}_p). \tag{$*4$}$$

(Note that we could replace $\mathcal{B}_i$ with $\mathcal{A}_i$, $i = p, q$, in $(*1)$ and $(*2)$, since the sections considered coincide.)

As we will show below, the glueing problem can be solved if these conditions are satisfied. Roughly speaking, they imply two things. First, they imply that around each of the points p and q, sections of $\mathcal{B}_p$ and $\mathcal{B}_q$ over $\hat{\dot{\mathbb{A}}}$ can be obtained as a product of a section over $\hat{\mathbb{A}}$ and a section over $\ddot{\mathbb{A}}$. This solves the glueing problem locally around each of the points p and q. Second, the fact that we can take sections over $\dot{\mathbb{A}}_p$ and $\dot{\mathbb{A}}_q$, and not only over

$\ddot{\mathbb{A}}$, in the left hand factors of the products in conditions (*3) and (*4) leads to a global solution of the glueing problem. To understand this intuitively, suppose that we have sections $a \in \mathcal{B}_p(\dot{\hat{\mathbb{A}}}_q)$ and $d \in \mathcal{B}_p(\dot{\hat{\mathbb{A}}}_p)$. Then we can write $a = b \cdot c$ with sections $b \in \mathcal{B}_p(\ddot{\mathbb{A}})$ and $c \in \mathcal{B}_p(\hat{\mathbb{A}}_q)$ according to condition (*2). This corresponds to glueing around q. Similarly, the glueing around p corresponds to $d = e \cdot f$ with $e \in \mathcal{B}_p(\dot{\mathbb{A}}_p)$ and $f \in \mathcal{B}_p(\hat{\mathbb{A}}_p)$. However, the glueing around p is in general not compatible with the glueing around q. As will be apparent from the proof of the proposition below, fitting the pieces together requires multiplying the sections b and e in $\mathcal{B}_p(\ddot{\mathbb{A}})$, and compatibility means that it must be possible to write the section $a = b \cdot c$ also as a product $a = (b \cdot e) \cdot c'$ for some section $c' \in \mathcal{B}_p(\hat{\mathbb{A}}_q)$. But since $e \in \mathcal{B}_p(\dot{\mathbb{A}}_p)$ is a section over $\dot{\mathbb{A}}_p$ and not only over $\ddot{\mathbb{A}}$, it also induces a section e over $\hat{\mathbb{A}}_q$. Putting $c' = e^{-1} \cdot c$, the requirement for compatibility can be met. Thus, once we know that the various local glueing problems can be solved, the fact that the subgroup schemes $\dot{\mathcal{A}}_q$ and $\dot{\mathcal{A}}_p$ from 4.3 extend over the points p and q respectively allows us to solve these local problems in a way that results in a global G-isomorphism $X \to Q$.

Proposition. *Suppose* (*1)–(*4) *are satisfied. Then X and Q are G-isomorphic.*

Proof. We show that the isomorphisms $\hat{\psi}_p$, $\hat{\psi}_q$ and $\ddot{\psi}$ can be multiplied by appropriate sections of $\mathcal{A}$ so that they satisfy the assumptions of 2.1. Consider the section $\hat{\psi}_q^{-1} \circ \ddot{\psi} \in \mathcal{A}(\dot{\hat{\mathbb{A}}}_q)$. Then, by Proposition 4.3:

$$\hat{\psi}_q^{-1} \circ \ddot{\psi} = \gamma \cdot \beta,$$

where $\gamma \in \mathcal{B}_q(\dot{\hat{\mathbb{A}}}_q)$ and $\beta \in \mathcal{B}_p(\dot{\hat{\mathbb{A}}}_q)$. By (*2) there are sections $\ddot{\beta} \in \mathcal{B}_p(\ddot{\mathbb{A}})$ and $\hat{\beta} \in \mathcal{B}_p(\hat{\mathbb{A}}_q)$ such that

$$\hat{\beta} \cdot \ddot{\beta} = \beta.$$

Similarly, the section $\hat{\psi}_p^{-1} \circ (\ddot{\psi} \cdot \ddot{\beta}^{-1}) \in \mathcal{A}(\dot{\hat{\mathbb{A}}}_p)$ can be written as

$$\hat{\psi}_p^{-1} \circ (\ddot{\psi} \cdot \ddot{\beta}^{-1}) = \delta \cdot \alpha$$

with $\delta \in \mathcal{B}_q(\dot{\hat{\mathbb{A}}}_p)$ and $\alpha \in \mathcal{B}_p(\dot{\hat{\mathbb{A}}}_p)$. By (*4) there are sections $\dot{\alpha} \in \mathcal{B}_p(\dot{\mathbb{A}}_p)$ and $\hat{\alpha} \in \mathcal{B}_p(\hat{\mathbb{A}}_p)$ such that

$$\hat{\alpha} \cdot \dot{\alpha} = \alpha.$$

Section $\dot{\alpha}$ induces by restriction sections of $\mathcal{B}_p(\ddot{\mathbb{A}})$ and $\mathcal{B}_p(\hat{\mathbb{A}}_q)$, which we also denote by $\dot{\alpha}$. Now consider the G-isomorphisms

$$\ddot{\phi} := \ddot{\psi} \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1} : \ddot{Q} \longrightarrow \ddot{X},$$
$$\hat{\phi}_p := \hat{\psi}_p \cdot \hat{\alpha} : \hat{Q}_p \longrightarrow \hat{X}_p,$$
$$\hat{\phi}_q := \hat{\psi}_q \cdot \hat{\beta} \cdot \dot{\alpha}^{-1} : \hat{Q}_q \longrightarrow \hat{X}_q.$$

Then we have

$$\begin{aligned}\hat{\phi}_q^{-1} \circ \ddot{\phi} &= \dot{\alpha} \cdot \hat{\beta}^{-1} \cdot (\hat{\psi}_q^{-1} \circ \ddot{\psi}) \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1} \\ &= \dot{\alpha} \cdot \hat{\beta}^{-1} \cdot \gamma \cdot \beta \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1}.\end{aligned}$$

Since $\mathcal{B}_q$ is a normal subgroup of $\mathcal{A}$, we can write this as

$$\begin{aligned}\hat{\phi}_q^{-1} \circ \ddot{\phi} &= \dot{\alpha} \cdot \hat{\beta}^{-1} \cdot \gamma \cdot \beta \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1} \\ &= \gamma' \cdot \dot{\alpha} \cdot \hat{\beta}^{-1} \cdot \beta \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1} \\ &= \gamma' \qquad (\text{since } \hat{\beta} \cdot \ddot{\beta} = \beta)\end{aligned}$$

for some section $\gamma' \in \mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q)$. Similarly, we have

$$\begin{aligned}\hat{\phi}_p^{-1} \circ \ddot{\phi} &= \hat{\alpha}^{-1} \cdot (\hat{\psi}_p^{-1} \circ \ddot{\psi}) \cdot \ddot{\beta}^{-1} \cdot \dot{\alpha}^{-1} \\ &= \hat{\alpha}^{-1} \cdot \delta \cdot \alpha \cdot \dot{\alpha}^{-1} \\ &= \delta' \cdot \hat{\alpha}^{-1} \cdot \alpha \cdot \dot{\alpha}^{-1} \\ &= \delta'\end{aligned}$$

for some section $\delta' \in \mathcal{B}_q(\hat{\dot{\mathbb{A}}}_p)$. We have thus reduced the sections β and α to the identity. In a completely analogous way, using $(*1)$ and $(*3)$, we can now multiply the new G-isomorphisms $\ddot{\phi}$, $\hat{\phi}_p$ and $\hat{\phi}_q$ by the appropriate local sections of $\mathcal{A}$ to reduce γ' and δ' to the identity section. $\square$

4.6. We use Section 4.3 to prove linearization for linear models of Type 1. In these cases, we only have to consider the group schemes associated with the N-action on the quadric $Q^H \subset V^H$ in the representation (V^H, N).

Proposition. *The automorphism group scheme associated to $Q^H \to \mathbb{A}$ satisfies the conditions* $(*)$ *in* 4.5. *Hence, by* 4.3, *all actions with a linear model of Type* 1 *are linearizable.*

Proof. We first consider the central subgroup $\mathcal{M} \cong \mathbb{A} \times \mathbb{C}^*$ and conditions $(*2)$ and $(*4)$. We may assume that $q = 0$. A section $\sigma \in \mathcal{M}(\hat{\dot{\mathbb{A}}}_q)$ is just a morphism $\hat{\dot{\mathbb{A}}}_q \to \mathbb{C}^*$, hence an (invertible) Laurent series $\sigma \in \mathbb{C}((z))^*$. It is clear that every Laurent series can be uniquely written as a product $r \cdot s$ with an invertible power series $r \in \mathbb{C}[[z]]^*$ and a unit $s \in \mathbb{C}[z, z^{-1}]$, if we impose $r(0) = 1$. Thus σ can be written as a product of sections in $\mathcal{M}(\hat{\mathbb{A}}_p)$ and in $\mathcal{M}(\dot{\mathbb{A}})$, hence $(*2)$ is satisfied. Similarly, $(*4)$ is satisfied as well, and we are left with showing the corresponding properties for the group scheme $\mathcal{B}_q$. We show first that $(*1)$ holds. Then we only have to consider the restriction $\dot{\mathcal{A}}_q = \mathcal{B}_q|_{\dot{\mathbb{A}}_q}$. We know that $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\mathbb{Z}_2} \mathbb{C}^*$, since $N_A \cong \mathbb{C}^*$. Assuming, as we may, that $p = 1$, we consider the double cover $\tilde{\mathbb{B}} \to \ddot{\mathbb{A}}$ given by $z \mapsto -z$, where $\tilde{\mathbb{B}} := \mathbb{B}\backslash\{-1,1,0\}$. Then $\mathbb{Z}_2$ acts on morphisms $\mathbb{C}^*(\tilde{\mathbb{B}})$ by sending $h(z)$ to $h(-z)^{-1}$, and the sections $\dot{\mathcal{A}}_q(\ddot{\mathbb{A}})$ are given by the fixed points $\mathbb{C}^*(\tilde{\mathbb{B}})^{\mathbb{Z}_2}$. Similarly, $\dot{\mathcal{A}}_q(\hat{\mathbb{A}}_p) = \mathbb{C}^*(\hat{\mathbb{B}}_p)^{\mathbb{Z}_2}$ and $\dot{\mathcal{A}}_q(\hat{\dot{\mathbb{A}}}_p) = \mathbb{C}^*(\hat{\dot{\mathbb{B}}}_p)^{\mathbb{Z}_2}$, where $\hat{\mathbb{B}}$ and $\hat{\dot{\mathbb{B}}}$ are given as usual. There is a surjective map $\mathbb{C}^*(\tilde{\mathbb{B}}) \times \mathbb{C}^*(\hat{\mathbb{B}}_p) \to \mathbb{C}^*(\hat{\dot{\mathbb{B}}}_p)$. Taking $\mathbb{Z}_2$-invariants $(*1)$ follows.

We are left with showing that $(*3)$ holds. We have seen in 4.3 that the connected component of the identity section $\mathcal{A}^0$ is isomorphic to $\mathbb{B} \times^{N/N^0} \mathcal{C}$, where $\mathcal{C} \cong \mathbb{A} \times (\mathbb{C}^*)^2$. Sections of $\mathcal{A}^0$ are just the N/N^0-invariant sections of $\mathcal{C}$. We assume that the fixed point of N/N^0 on $\mathbb{B}$ is the point 0, hence that $\hat{\dot{\mathbb{B}}} = \operatorname{Spec}\mathbb{C}((z))$ if we let $\mathbb{B} = \operatorname{Spec}\mathbb{C}[z]$. Since $\mathcal{C}$ is constant around 0, an element $\hat{\dot{\alpha}} \in \mathcal{C}(\hat{\dot{\mathbb{B}}})$ is given by two units $h(z), k(z)$ in $\mathbb{C}((z))^*$:

$$\hat{\dot{\alpha}} = (h(z), k(z)).$$

It follows from the description of the N-action given in 4.3 that the generator $\gamma \in N/N^0$ acts on such a section by

$$\gamma \cdot \hat{\dot{\alpha}} = (h(-z), k(-z)^{-1}).$$

The sections of the form $(h(z^2), 1)$ are γ-invariant and give all sections of the central subgroup $\mathcal{M}$ over $\hat{\dot{\mathbb{A}}}_q$. Sections of $\mathcal{B}_q$ over $\hat{\dot{\mathbb{A}}}_q$ are given by the fixed points $\mathbb{C}^*(\hat{\dot{\mathbb{B}}})^{N/N^0}$, i.e., by Laurent series $k(z)$ for which $k(-z)^{-1} = k(z)$. Similarly, $\mathcal{B}_q(\dot{\mathbb{A}}) = \mathbb{C}[z, z^{-1}]^{*N/N^0}$ and $\mathcal{B}_q(\hat{\mathbb{A}}_q) = \mathbb{C}[[z]]^{*N/N^0}$. As in the case for the central subgroup above the multiplication induces a surjective map

$$\mathbb{C}[z, z^{-1}]^* \times \mathbb{C}[[z]]^* \to \mathbb{C}((z))^*.$$

Taking N/N^0-invariants the claim follows. □

4.7. We make a few remarks concerning the proof of Proposition 4.6. To prove $(*2)$, $(*3)$ and $(*4)$ in 4.6, we used certain properties of the sections of group schemes of the form $\mathcal{C} = \dot{\mathbb{B}} \times^{\Gamma} \mathbb{C}^*$ over $\dot{\mathbb{A}}$ (for $(*2)$ and $(*4)$ the group Γ involved was trivial). Roughly speaking, these properties are that sections of $\mathcal{C}$ over the (formal) punctured disc at the origin can be obtained as a product of a section over the punctured disc which extends as the identity over the origin, and a section over $\dot{\mathbb{A}}$. Since sections of $\mathcal{C}$ are just Γ-invariant morphisms to $\mathbb{C}^*$ (where $\gamma \in \Gamma$ acts on a morphism in the usual way by ${}^{\gamma}\sigma = \gamma\cdot\sigma\cdot\gamma^{-1}$), this means that $\mathbb{C}^*(\hat{\dot{\mathbb{B}}})^{\Gamma} = \mathbb{C}^*(\dot{\mathbb{B}})^{\Gamma}\cdot\mathbb{C}^*(\hat{\mathbb{B}})^{\Gamma}_1$, where the subscript 1 denotes those sections which induce the identity in the fiber over the origin. That is, $\mathbb{C}^*$ has the *decomposition property* , see [Kr–S,V.0.5]. In fact, every algebraic group has this property: Let the finite group Γ act on the affine line $\mathbb{B}$ and on the algebraic group M via group automorphisms. For μ_1, $\mu_2 \in M(\hat{\mathbb{B}})$ write $\mu_1 = \mu_2 + O(t^r)$ if $\mu_1\mu_2^{-1}$ is tangent to I to order

r at 0, where I denotes the constant map to the identity of M. Define $M(\hat{\mathbb{B}})_r \coloneqq \{\mu \in M(\hat{\mathbb{B}}) \mid \mu = I + \mathrm{o}(t^r)\}$, and set $M(\hat{\mathbb{B}})_r^\Gamma := M(\hat{\mathbb{B}})_r \cap M(\hat{\mathbb{B}})^\Gamma$. Then

$$M(\dot{\hat{\mathbb{B}}})^\Gamma = M(\dot{\mathbb{B}})^\Gamma \cdot M(\hat{\mathbb{B}})_1^\Gamma,$$

see [Kr–S, V.2.4]. Moreover, if M is semisimple or unipotent then M has the *approximation property*:

$$M(\dot{\hat{\mathbb{B}}})^\Gamma = M(\dot{\mathbb{B}})^\Gamma \cdot M(\hat{\mathbb{B}})_r^\Gamma \qquad \forall r \geq 1,$$

see [Kr–S,V.3.5]. This will be used in 4.10.

4.8. We have now reached the final stretch and show linearization for linear models of Type 2. Unfortunately, the map $\mathrm{res}_V : \Delta(V)^G \to \Delta(V^H)^N$ of 4.2 is not always an isomorphism in this case, and we can't restrict the problem as we did for linear models of Type 1.

Conditions $(*1)$ and $(*2)$ in 4.5 are conditions on the group schemes $\dot{\mathcal{A}}_q$ and $\dot{\mathcal{A}}_p$ of Proposition 4.3. Since we restrict our attention to the connected component $\mathcal{A}^0$ of $\mathcal{A}$ and hence to the connected component of the identity of the group N, we can assume that both $N_A \subset N$ and $N_B \subset N$ are either isomorphic to $\mathbb{C}^*$ or to SL_2. (The only other possibility would be $\mathbb{Z}_2$, see 1.5 and the proof of Proposition 4.3.) If N_A or N_B are isomorphic to $\mathbb{C}^*$, the conditions follow in exactly the same way as condition $(*1)$ was proved for the group scheme $\dot{\mathcal{A}}_q$ in Proposition 4.6. If N_A or N_B are isomorphic to SL_2, the corresponding group scheme $\dot{\mathcal{A}}_q$ or $\dot{\mathcal{A}}_p$ is constant, because the automorphism group of SL_2 is connected. This was shown in the proof of Proposition 4.4. The conditions then follow from the decomposition property for algebraic groups (see 4.7). If the group schemes $\mathcal{B}_q$ and $\mathcal{B}_p$ are constant, i.e., isomorphic to $\mathbb{A} \times N_A$ and $\mathbb{A} \times N_B$ respectively, the other two conditions are satisfied for the same reason. For example, this holds for all linear models in Table 2 in the Appendix.

4.9. It remains to prove that $(*3)$ and $(*4)$ hold when N_A and N_B are isomorphic to $\mathbb{C}^*$. We will show that $(*3)$ and $(*4)$ are conditions on the degrees of the generating covariants of the linear model (V, G). Recall that $\mathcal{B}_q$ is the closure of $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\Gamma_B} N_A$ in $\mathcal{A}$, while $\mathcal{B}_p$ is the closure of $\dot{\mathcal{A}}_p \cong \dot{\mathbb{A}}_p \times N_B$ in $\mathcal{A}$ (see Proposition 4.3). In the sequel, we will make all the arguments for the group scheme $\mathcal{B}_q$. However, replacing the group Γ_B in these arguments by the trivial group, it will be clear that the same arguments are valid for the group scheme $\mathcal{B}_p$.

The isomorphism $\dot{\mathcal{A}}_q \cong \dot{\mathbb{B}} \times^{\Gamma_B} N_A$ induces an isomorphism of sections

$$\hat{\varphi} : N_A(\dot{\hat{\mathbb{B}}})^{\Gamma_B} \cong \mathcal{B}_q(\dot{\hat{\mathbb{A}}}_q),$$

and the restriction of $\hat{\varphi}$ to $N_A(\dot{\mathbb{B}})^{\Gamma_B}$ is an isomorphism $N_A(\dot{\mathbb{B}})^{\Gamma_B} \cong \mathcal{B}_q(\dot{\mathbb{A}}_q)$. The sticky point is to see whether $\hat{\varphi}$ maps the sections $N_A(\hat{\mathbb{B}})^{\Gamma_B} \subset N_A(\dot{\hat{\mathbb{B}}})^{\Gamma_B}$ to $\mathcal{B}_q(\hat{\mathbb{A}}_q)$, i.e., to the sections in $\mathcal{B}_q(\dot{\hat{\mathbb{A}}}_q)$ that extend over the point q.

Lemma. *Suppose that* $\hat{\dot{\varphi}}(N_A(\hat{\mathbb{B}})_1^{\Gamma_B}) \subset \mathcal{B}_q(\hat{\mathbb{A}}_q)$ *in* $\mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q)$. *Then* (*3) *is satisfied. By symmetry, an analogous statement is true for* (*4).

Proof. Let $b \in \mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q)$, and let $a := \hat{\dot{\varphi}}^{-1}(b)$. By the decomposition property for N_A, see 4.7, there are sections $\dot{a} \in N_A(\dot{\mathbb{B}})^{\Gamma_B}$ and $\hat{a} \in N_A(\hat{\mathbb{B}})_1^{\Gamma_B}$ with $a = \dot{a} \cdot \hat{a}$. Clearly $\dot{b} := \hat{\dot{\varphi}}(\dot{a}) \in \mathcal{B}_q(\dot{\mathbb{A}}_q)$, and by assumption $\hat{b} := \hat{\dot{\varphi}}(\hat{a}) \in \mathcal{B}_q(\hat{\mathbb{A}}_q)$. It follows that $b = \dot{b} \cdot \hat{b}$, hence that $\mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q) \subset \mathcal{B}_q(\dot{\mathbb{A}}_q) \cdot \mathcal{B}_q(\hat{\mathbb{A}}_q)$. □

4.10. If N_A or N_B is isomorphic to SL_2, it is enough that $\hat{\dot{\varphi}}(N_A(\hat{\mathbb{B}})_r^{\Gamma_B}) \subset \mathcal{B}_q(\hat{\mathbb{A}}_q)$ for some $r \geq 1$, because SL_2 has the approximation property, see (4.7).

Proposition. *Suppose that* $N_A \cong \mathrm{SL}_2$. *Then* (*3) *is satisfied. By symmetry,* (*4) *is satisfied if* $N_B \cong \mathrm{SL}_2$.

Proof. Note that $\mathcal{O}(\hat{\dot{\mathbb{B}}}) \cong \mathbb{C}((z)) \cong \mathcal{O}(\hat{\dot{\mathbb{A}}}_q)$ are topological fields with $\mathcal{O}(\hat{\mathbb{B}})_r \cong z^r\mathbb{C}[[z]] \cong \mathcal{O}(\hat{\mathbb{A}}_q)_r$ as fundamental systems of neighborhoods of 0. The groups $N_A(\hat{\dot{\mathbb{B}}})^{\Gamma_B}$ and $\mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q)$ inherit the corresponding topologies, and it is easy to see that the map $\hat{\dot{\varphi}}$ is continuous in these topologies (recall that $\hat{\dot{\varphi}}$ is induced by the isomorphism φ in the proof of Proposition 4.3; cf. also [Kr–S, VI.1.17]). It follows that there is a number $r_0 \geq 1$ such that

$$\hat{\dot{\varphi}}(N_A(\hat{\mathbb{B}})_{r_0}^{\Gamma_B}) \subset \mathcal{B}_q(\hat{\mathbb{A}}_q).$$

Let $b \in \mathcal{B}_q(\hat{\dot{\mathbb{A}}}_q)$ and $a := \hat{\dot{\varphi}}^{-1}(b)$ be the sections as in the proof of Lemma 4.9. Since SL_2 has the approximation property, see 4.7, we can now write $a = \dot{a} \cdot \hat{a}$ with $\dot{a} \in N_A(\dot{\mathbb{B}})^{\Gamma_B}$ and $\hat{a} \in N_A(\hat{\mathbb{B}})_{r_0}^{\Gamma_B}$. Then $b = \hat{\dot{\varphi}}(\dot{a}) \cdot \hat{\dot{\varphi}}(\hat{a}) \in \mathcal{B}_q(\dot{\mathbb{A}}_q) \cdot \mathcal{B}_q(\hat{\mathbb{A}}_q)$. □

4.11. We are left with the cases where N_A or N_B is isomorphic to $\mathbb{C}^*$. We work towards showing that, taking differentials, the condition in Lemma 4.9 translates to a condition on the degree of the generating covariants of the linear model. Let $(V, G) = (V_1, G) \oplus (V_2, G)$ be the linear model, and let f and g be the invariants in $\mathcal{O}(V_1)^G$ and $\mathcal{O}(V_2)^G$ defining the invariant quadric $Q = \{(v_1, v_2) \in V \mid f(v_1) + g(v_2) = 1\}$ as in 4.3.

We denote the covariants of the linear model by $\Delta(V)^G$: they are the G-invariant vector fields on V, or, equivalently, the G-equivariant derivations of $\mathcal{O}(V)$. Let $\Delta(V)^G_{f,g}$ be the elements in $\Delta(V)^G$ annihilating the $\mathcal{O}(V)^G = \mathbb{C}[f, g]$, i.e., the G-invariant vector fields along the fibers of the quotient map. Note that $\Delta(V)^G$ is a free graded $\mathcal{O}(V)^G$-module of rank $\dim V^H$, see [S2]. Let $x_1, \ldots, x_n$ be coordinates on $V_1 \subset V$ such that $f = x_1^2 + \ldots + x_n^2$, and let $y_1, \ldots, y_m$ be coordinates on $V_2 \subset V$ such that $g = y_1^2 + \ldots + y_m^2$.

Lemma.

(1) *As* $\mathcal{O}(V)^G$*-module,* $\Delta(V)^G$ *decomposes as*

$$\Delta(V)^G = \Delta_{f,g}(V)^G \oplus \mathbb{C}[f, g] \cdot A_1 \oplus \mathbb{C}[f, g] \cdot A_2,$$

where $A_1 := \sum_{i=1}^{n} x_i \cdot \frac{\partial}{\partial x_i}$ *and* $A_2 := \sum_{j=1}^{m} y_j \cdot \frac{\partial}{\partial y_j}$ *are the Euler vector fields on* V_1 *and* V_2.

(2) *Let* $\mathfrak{n} := \operatorname{Lie} N$. *Then* $\mathfrak{n} = \Delta_{f,g}(V)^G|_F$.

Proof. (1) is left as an exercise. (One checks that for every $C \in \Delta(V)^G$, $C(f)$ and $C(g)$ are multiples of f and g respectively, and then uses that $A_1(f) = 2f$ and $A_2(g) = 2g$, while $A_1(g) = A_2(f) = 0$.) Since $\Delta_{f,g}(V)^G$ consists of vectorfields along the fibers of the quotient map we have $\Delta_{f,g}(V)^G|_F \subseteq \mathfrak{n}$. But $\dim \mathfrak{n} = \dim V^H - 2 = \operatorname{rk}_{\mathbb{C}[f,g]} \Delta_{f,g}(V)^G$ by (1), and the claim follows. $\square$

4.12. We give $\mathcal{O}(\hat{Q}_q)$ the filtration $\{\mathcal{O}(\hat{Q}_q)_j\}$ where $\mathcal{O}(\hat{Q}_q)_j = \{h \in \mathcal{O}(\hat{Q}_q) \mid h \in \mathbf{m}^j\}$, where $j \geq 0$ and where $\mathbf{m}$ is the maximal ideal belonging to a point $w = (v, 0) \in V = V_1 \oplus V_2$ in the closed orbit in $\pi_Q^{-1}(q)$ with isotropy group B. We give $\mathcal{O}(\hat{\mathbb{A}}_q)$ the induced filtration $\{\mathcal{O}(\hat{\mathbb{A}}_q)_j = \mathcal{O}(\hat{Q}_q)_j^G\}$.

It follows from the last section that there is a homogeneous generator D of $\Delta_{f,g}(V)^G$ whose restriction to F is a generator of the 1-dimensional subspace $\mathfrak{n}_A \subset \mathfrak{n}$. Let the variety $\mathcal{X}$ over $\hat{\mathbb{A}}_q$ be defined as $\mathcal{X} := \hat{\mathbb{A}}_q \times \mathbb{C} \cdot D$. Since D is a vectorfield along the fiber F, the sections $\mathcal{X}(\hat{\mathbb{A}}_q)$ induce G-derivations of $\mathcal{O}(\hat{Q}_q)$. For $r \geq 0$, we define $\mathcal{X}(\hat{\mathbb{A}}_q)_r := \{C \in \mathcal{X}(\hat{\mathbb{A}}_q) \mid C(\mathcal{O}(\hat{Q}_q)_j) \subseteq \mathcal{O}(\hat{Q}_q)_{j+r} \forall j\}$. Confusing elements of $\mathcal{B}_q(\hat{\mathbb{A}}_q)$ with the G-automorphisms of $\mathcal{O}(\hat{Q}_q)$ that they induce, we define $\mathcal{B}_q(\hat{\mathbb{A}}_q)_r := \{\hat{\alpha} \in \mathcal{A} \mid \hat{\alpha}(y_i) = y_i + \mathcal{O}(\hat{Q}_q)_{r+1}\}$ as the elements in $\mathcal{B}_q(\hat{\mathbb{A}}_q)$ which are tangent to the identity to order r at w (where the y_i's are a system of generators of $\mathbf{m}$). If $C \in \mathcal{X}(\hat{\mathbb{A}}_q)$, we define $\exp(C)$ to be the usual series $I + C + \frac{C \circ C}{2} + \dots$, where I is the identity and $\circ$ denotes the composition of derivations. Similarly we define the logarithm of an endomorphism of $\mathcal{O}(\hat{Q}_q)$.

Proposition. *The exponential* $C \mapsto \exp(C)$ *gives an isomorphism (with inverse* $\log$*) of* $\mathcal{X}(\hat{\mathbb{A}}_q)_r$ *with* $\mathcal{B}_q(\hat{\mathbb{A}}_q)_r$ *for* $r \geq 1$.

Proof. (Cf. [Kr–S,VI.1.6]). Let $C \in \mathcal{X}(\hat{\mathbb{A}}_q)_r$, and consider an isotypic component $\mathcal{O}(\hat{Q}_q)^G_{(\omega)} = \mathcal{O}(\hat{\mathbb{A}}_q) \otimes S_{(\omega)}$, see 4.2. Then $S_{(\omega)} \cong kV^*_{(\omega)}$, where $V_{(\omega)}$ is an irreducible component of V. Let l denote the maximal number such that $h \in \mathcal{O}(\hat{Q}_q)_l$ for $h \in S_{(\omega)}$. Then C preserves $\mathcal{O}(\hat{Q}_q)^G_{(\omega)}$, and the action of C on $\mathcal{O}(\hat{Q}_q)^G_{(\omega)}$ is given by a matrix $(a_{ij}) \in M_k(\mathcal{O}(\hat{\mathbb{A}}_q))$. The action of C^p corresponds to the p-th power $(a_{ij}^{(p)})$ of (a_{ij}), where the $a_{ij}^{(p)}$ lie in $\mathcal{O}(\hat{\mathbb{A}}_q)_{\max(0,pr-l+1)}$. Since $r \geq 1$, the series $(a_{ij}^{(p)})$ converges in the $\mathbf{m}$-adic topology, i.e., $\exp(C)$ gives an automorphism of $\mathcal{O}(\hat{Q}_q)^G_{(\omega)}$. It follows that $\exp(C)(h)$ exists for all $h \in \mathcal{O}(\hat{Q}_q)$. It is clear that the corresponding automorphism of $\mathcal{O}(\hat{Q}_q)$ is an element of $\mathcal{B}_q(\hat{\mathbb{A}}_q)_r$.

Conversely, let $\hat{\alpha} \in \mathcal{B}_q(\hat{\mathbb{A}}_q)_r$, $r \geq 1$. On $\mathcal{O}(\hat{Q}_q)_{(\omega)}$, $\hat{\alpha}$ corresponds to a matrix $(a_{ij}) \in \mathrm{GL}_k(\mathcal{O}(\hat{Q}_q))$, and arguments as above show that the series

$\log((a_{ij}))$ converges to an element of $M_k(\mathcal{O}(\hat{Q}_q))$ which sends $\mathcal{O}(\hat{Q}_q)_{(\omega)} \cap \mathcal{O}(\hat{Q}_q)_r$ to $\mathcal{O}(\hat{Q}_q)_{(\omega)} \cap \mathcal{O}(\hat{Q}_q)_{r+s}$. The claim follows easily. □

4.13. Together with Corollary 3.5 and Proposition 4.6 the next result finishes the proof of Theorem 1.

Proposition. *If the linear model for a given action is of Type 2, then the action is linearizable.*

Proof. Recall first the setup: we assume that the quotient $Q = \{(v_1, v_2) \in V = V_1 \oplus V_2 \mid f(v_1) + g(v_2) = 1\} \to \mathbb{A}$ is given by the restriction of g to Q. Then the exceptional points in $\mathbb{A}$ are $p = 1$ and $q = 0$, and the slice representations are (V_1, A) and (V_2, B). For $c \neq 0, 1$, we set $F_1 = f^{-1}(1-c) \subset V_1$ and $F_2 = g^{-1}(c) \subset V_2$. Then $F = F_1 \times F_2$ is a generic G-orbit in Q.

Consider again $\mathfrak{n}_A = \mathbb{C} \cdot D$, with $D \in \Delta(V)^G_{f,g}$ from 4.12. Here and in the sequel D also denotes the restriction $D|_F$. As in 4.3, the N_A-action on $F_1 \times F_2$ is induced by the action of N_A on the generic orbit F_1 of the slice representation (V_1, A). It follows that D must be a covariant of type V_1. If $d_2 = \deg_{V_2} D$ is the degree of D in the coordinates of V_2, the Γ_B-action on N_A therefore induces an action on $\mathfrak{n}_A$ given by ${}^{\gamma}D = \gamma^{-d_2} \cdot D$. Γ_B acts on $\hat{\mathbb{B}}$ as usual, and we denote by $\mathfrak{n}_A(\hat{\mathbb{B}})^{\Gamma_B}_r$ those Γ_B-invariant morphism $\hat{\mathbb{B}} \to \mathfrak{n}_A$ which vanish to order r at 0, cf. 4.7. The exponential $\exp : \mathfrak{n}_A \to N$ is Γ_B-equivariant, and since it is a local analytic isomorphism, it easily follows that it induces isomorphisms $\exp : \mathfrak{n}_A(\hat{\mathbb{B}})^{\Gamma_B}_r \xrightarrow{\sim} N(\hat{\mathbb{B}})^{\Gamma_B}_r$ for all $r \geq 1$. On the other hand, there are isomorphisms $\exp : \mathcal{X}(\hat{\mathbb{A}}_q)_r \to \mathcal{B}_q(\hat{\mathbb{A}}_q)_r$ for all $r \geq 1$ by 4.12. Consider now the isomorphism of sections

$$\hat{\dot{\varphi}} : N_A(\hat{\dot{\mathbb{B}}})^{\Gamma_B} \to \mathcal{B}_q(\hat{\ddot{\mathbb{A}}}_q)$$

(see 4.9). Its "differential" is an isomorphism

$$\varphi_\natural : \mathfrak{n}_A(\hat{\dot{\mathbb{B}}})^{\Gamma_B} \to \mathcal{X}(\hat{\ddot{\mathbb{A}}}_q),$$

which is defined by

$$(\varphi_\natural C)(h) \circ \hat{\dot{\varphi}} = C(h \circ \hat{\dot{\varphi}}) \qquad (*)$$

for $h \in \mathcal{O}(\hat{\ddot{Q}}_q)$, $C \in \mathfrak{n}_A(\hat{\dot{\mathbb{B}}})^{\Gamma_B}$. It then follows from the various isomorphisms exp above that the condition in Lemma 4.9 is satisfied if

$$\varphi_\natural(\mathfrak{n}_A(\hat{\mathbb{B}})^{\Gamma_B}_1) \subset \mathcal{X}(\hat{\mathbb{A}}_q)_1.$$

We translate this condition into a condition on the degree of the generating covariants in $\Delta_{f,g}(V)^G$. Recall that $\hat{\dot{\varphi}}$ is induced by the G-isomorphism $\psi : Y \times^{\Gamma_A \times \Gamma_B} (F_1 \times F_2) \to \ddot{Q}$ that was constructed in the proof of Proposition 4.3. Here $Y = \dot{\mathbb{D}} \times_{\ddot{\mathbb{A}}} \dot{\mathbb{B}}$, where $\mathbb{D}$ and $\mathbb{B}$ are copies of the affine line, and where $\dot{\mathbb{D}} \to \ddot{\mathbb{A}}$ is given by $t \mapsto -t^2(1-c)+1$, and $\dot{\mathbb{B}} \to \ddot{\mathbb{A}}$ is given by $z \mapsto cz^2$. The isomorphism ψ is then given by $[(t,z),(v_1,v_2)] \mapsto (tv_1, zv_2)$. For $(t,z) \in Y$ we have $t = \pm\sqrt{\frac{1-cz^2}{1-c}}$. It follows that over $\hat{\dot{\mathbb{A}}}_q$, the Γ_A-action on $\dot{\mathbb{B}}$ and F_1 cancels out to give a G-isomorphism over $\hat{\dot{\mathbb{A}}}_q$:

$$\hat{\dot{\mathbb{B}}} \times^{\Gamma_B} (F_1 \times F_2) \to \hat{\dot{Q}}_q,$$
$$[z,(v_1,v_2)] \mapsto (k(z) \cdot v_1, zv_2),$$

where $k(z) = \frac{1-cz^2}{1-c}$.This isomorphism induces the isomorphism of sections $\hat{\dot{\varphi}}$.

If $h \in \mathcal{O}(\hat{\dot{Q}}_q)_s$ is the restriction of a regular function $h \in \mathcal{O}(V)$ which is homogenous of degrees s' and s in V_1 and V_2, then

$$D(h)(\hat{\dot{\varphi}}[z,v_1,v_2]) = z^{s+d_2} \cdot k(z)^{s'+d_1-1} \cdot D(h|_F)(v_1,v_2),$$

where $d_1 = \deg_{V_1} D$ and $d_2 = \deg_{V_2} D$ are the degrees of D in the coordinates of V_1 and V_2. This follows from the fact that D is a covariant of type V_1. Thus, $D(h) \circ \hat{\dot{\varphi}} = z^{s+d_2} \cdot k(z)^{s'+d_1-1} \cdot D(h|_F)$. On the other hand, we have $(h \circ \hat{\dot{\varphi}})[z,v_1,v_2] = z^s \cdot k(z)^{s'} \cdot h(v_1,v_2)$, and it follows that

$$D(h \circ \hat{\dot{\varphi}})[z,v_1,v_2] = z^s \cdot k(z)^{s'} \cdot D(h|_F)(v_1,v_2),$$

i.e., that $D(h \circ \hat{\dot{\varphi}}) = z^s \cdot k(z)^{s'} \cdot D(h|_F)$. Using (*), we obtain for $z^r \cdot D \in \mathfrak{n}_A(\hat{\mathbb{B}})$

$$\varphi_{\mathfrak{h}}(z^r \cdot D) = z^{r-d_2} \cdot k(z)^{-d_1+1} \cdot D.$$

Suppose $z^r \cdot D \in \mathfrak{n}_A(\hat{\mathbb{B}})_1^{\Gamma_B}$. Then $0 < r \equiv d_2(2)$. Hence if $d_2 \leq 2$, then $r - d_2 \geq 0$. Since $k(z)$ is a unit in $\mathcal{O}(\hat{\mathbb{B}})$ $(k(0) \neq 0)$, it easily follows that in this case $\varphi_{\mathfrak{h}}(\mathfrak{n}_A(\hat{\mathbb{B}})_1^{\Gamma_B}) \subset \mathcal{X}(\hat{\mathbb{A}})_1$. To get condition (∗3) it is therefore enough to check that the V_2-degree of all generating covariants is ≤ 2. Using exactly the same arguments as for the group scheme $\mathcal{B}_q$, but with the group Γ_B replaced by the trivial group, it follows that condition (∗4) for the group scheme $\mathcal{B}_p$ is satisfied if the V_1-degree of all generating covariants is ≤ 2. These conditions on the degree of covariants have to be checked for those linear models of Type 1 for which the group schemes $\mathcal{B}_q$ or $\mathcal{B}_p$ are not constant and have generic fibers isomorphic to $\mathbb{C}^*$ (see 4.8), i.e., for those models in Table 3 for which the total degree d of the generating covariants is listed. In all these cases, $d \leq 3$. Now $d = d_1 + d_2$, and one easily verifies that if $d = 3$ then $d_1 \geq 1$ and $d_2 \geq 1$. This finishes the proof. □

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Appendix

Here we list all linear models for actions with 1-dimensional quotient as defined in 1.2. We repeat the tables obtained in [D, §5] with some additional information. Table 1 contains all orthogonal representations of connected reductive groups with 1-dimensional quotient. Tables 2–5 contain the orthogonal representations of connected reductive groups with 2-dimensional quotient. Table 2 contains the representations of the form $(V = V_1 \oplus V_2, G = G_1 \times G_2)$, where (V_i, G_i), $i = 1, 2$, are representations from Table 1, and where G_1 acts trivially on V_2 and G_2 acts trivially on V_1. Table 3 contains the mixed cases, which are as above but without the assumption that G_1 and G_2 act trivially on V_2 and V_1 respectively. Tables 4 and 5 contain those representations which are complexifications of real irreducible representations. Table 4 contains the ones which remain irreducible after complexification, the others appear in Table 5.

In the tables G will denote the group that acts according to standard notation. In the column V we list the representation using the notation in [Li]: ω_i denotes the irreducible representation corresponding to the fundamental weight ω_i, $n\omega_i$ the one with the highest weight $n\omega_i$ $(n \in \mathbb{N})$. If G has more than one simple factor, the fundamental weights of the second factor (and the corresponding representation) are denoted by ω_i', the one of the third factor by ω_i''. ω_i^* denotes the dual representation of ω_i, etc. If $a \in \mathbb{Z}$, then Σ_a denotes the 1-dimensional representation of $\mathbb{C}^*$ with weight a.

Under H we list the principal isotropy group, and under A and B in Tables 2–5 the exceptional isotropy groups of the G-action on Q_V (cf. Proposition 1 in 1.3).

Under N we list the G-automorphism group $N_G(H)/H$ of the generic fiber G/H of the quotient map, i.e., N is the generic fiber of the associated automorphism group scheme $\mathcal{A}_V^G$. In Table 4, $\mathcal{D}_n$ denotes the dihedral group of order $2n$. N is either computed directly or with the help of the Luna-Richardson theorem (cf. 3.4).

In Table 2, (ω, C) and (ω', D) are representations from Table 1, Σ and Σ' denote 1-dimensional trivial representation of C and D respectively. C' and D' denote the principal isotropy groups of (ω, C) and (ω', D); N_C and N_D denote the C- resp. D-automorphism groups of C/C' and D/D'.

In Table 3 we list under d the (total) degree of the generating covariants of $\Delta_{f,g}(V)^G$. To compute this degree one computes the generators of $\Delta(V)^G$ as follows: By 4.2 there is a G-stable subspace $T \subset \mathcal{O}(V)$ which is a graded, free module over $\mathcal{O}(V)^G$ and such that the multiplication induces a G-isomorphism $\mathcal{O}(V)^G \otimes T \xrightarrow{\sim} \mathcal{O}(V)$. If V_λ is an irreducible subspace of V, then generators of the equivariant maps from V to V_λ correspond to copies of V_λ^* in T. By Theorem 1.1 in [S1], V_λ^* occurs in T with multiplicity $m_\lambda = \dim V_\lambda^H$. Given m_λ, one computes the decomposition of the homogeneous components $\mathcal{O}(V)_j$, $j \geq 1$, of $\mathcal{O}(V)$ as G-module until one comes across the required number of copies of V_λ^* in T. A blank entry in the column d indicates that the associated group scheme is constant. An entry $*$ indicates that, although the group scheme is not constant, one doesn't need to know these degrees because of the approximation property for SL_2 (see 4.10). Note that all the associated group schemes $\mathcal{B}_q$ and $\mathcal{B}_p$ of 4.5 are constant for linear models in Table 2.

Table 1

G	V	H	N
$\mathbb{C}^*$	$\Sigma_a \oplus \Sigma_{-b},\ a, b > 0$	finite	$\mathbb{C}^*$
$A_n\ (n \geq 1)$	$\omega_1 \oplus \omega_1^*$	A_{n-1}	$\mathbb{C}^*$
$\mathbb{C}^* \times A_n\ (n \geq 1)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*,\ a > 0$	$\mathbb{C}^* \times A_{n-1}$	$\mathbb{C}^*$
$B_n\ (n \geq 2)$	ω_1	D_n	$\mathbb{Z}_2$
$D_n\ (n \geq 3)$	ω_1	B_{n-1}	$\mathbb{Z}_2$
$C_n\ (n \geq 2)$	$\omega_1 \oplus \omega_1^*$	C_{n-1}	A_1
$\mathbb{C}^* \times C_n\ (n \geq 2)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')\ ^*,\ a > 0$	$\mathbb{C}^* \times C_{n-1}$	$\mathbb{C}^*$
$A_1 \times C_n\ (n \geq 2)$	$\omega_1 \otimes \omega_1'$	$A_1 \times C_{n-1}$	$\mathbb{Z}_2$
$A_1 \times A_1$	$\omega_1 \otimes \omega_1'$	A_1	$\mathbb{Z}_2$
A_1	ω_2	$\mathbb{C}^*$	$\mathbb{Z}_2$
A_3	ω_2	C_2	$\mathbb{Z}_2$
B_3	ω_3	G_2	$\mathbb{Z}_2$
B_4	ω_4	B_3	$\mathbb{Z}_2$
C_2	ω_2	$A_1 \times A_1$	$\mathbb{Z}_2$
G_2	ω_1	A_2	$\mathbb{Z}_2$

Table 2

G	V	H	A	B	N
C	$\omega \oplus \Sigma$	C'	C	C	N_C
$C \times D$	$\omega \oplus \omega'$	$C' \times D'$	$C' \times D$	$C \times D'$	$N_C \times N_D$

Table 4

G	V	H	A	B	N
A_1	$4\omega_1$	$\mathcal{D}_4$	$\mathbb{C}^* \ltimes \mathbb{Z}_2$	$\mathbb{C}^* \ltimes \mathbb{Z}_2$	$\mathcal{D}_3$
A_2	$\omega_1\omega_2$	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$	$\mathcal{D}_3$
B_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$	$\mathcal{D}_4$
D_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
C_3	ω_2	$(A_1)^3$	$C_2 \times A_1$	$A_1 \times C_2$	$\mathcal{D}_3$
F_4	ω_1	D_4	B_4	B_4	$\mathcal{D}_3$
G_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$	$\mathcal{D}_6$
$A_1 \times A_1$	$3\omega_1 \otimes \omega_1'$	$\mathcal{D}_4$	$\mathbb{C}^* \ltimes \mathbb{Z}_2$	$\mathbb{C}^* \ltimes \mathbb{Z}_2$	$\mathcal{D}_6$
$C_2 \times C_m\ (m \geq 2)$	$\omega_1 \otimes \omega_1'$	$(A_1)^2 \times C_{m-2}$	$C_2 \times C_{m-2}$	$(A_1)^2 \times C_{m-1}$	$\mathcal{D}_4$

Table 3

G	V	H	A	B	N	d
A_3	$\omega_1 \oplus \omega_1^* \oplus \omega_2$	A_1	A_2	C_2	$A_1 \times \mathbb{C}^*$	$1, 2, 2, 3$
$\mathbb{C}^* \times A_3$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^* \oplus \omega_2'$	$\mathbb{C}^* \times A_1$	$\mathbb{C}^* \times A_2$	$\mathbb{C}^* \times C_2$	$\mathbb{C}^* \times \mathbb{C}^*$	$1, 3$
B_3	$\omega_1 \oplus \omega_3$	A_2	G_2	A_3	$\mathbb{C}^* \rtimes \mathbb{Z}_2$	2
D_4	$\omega_1 \oplus \omega_3$	G_2	B_3	B_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
$A_1 \times C_n$ $(n \geq 1)$	$2\omega_1 \oplus (\omega_1 \otimes \omega_1')$	$\mathbb{C}^* \times C_{n-1}$	$A_1 \times C_{n-1}$	$\mathbb{C}^* \times C_n$	$\mathbb{C}^* \rtimes \mathbb{Z}_2$	3
$A_1 \times C_n$ $(n \geq 1)$	$\omega_1 \oplus \omega_1^* \oplus (\omega_1 \otimes \omega_1')$	C_{n-1}	$A_1 \times C_{n-1}$	C_n	$A_1 \times A_1$	$*$
$A_1 \times A_1 \times C_n$ $(n \geq 1)$	$(\omega_1 \otimes \omega_1') \oplus (\omega_1' \otimes \omega_1'')$	$A_1 \times C_{n-1}$	$(A_1)^2 \times C_{n-1}$	C_n	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
$\mathbb{C}^* \times A_1 \times C_n$ $(n \geq 1)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^* \oplus (\omega_1' \otimes \omega_1'')$	$\mathbb{C}^* \times C_{n-1}$	$\mathbb{C}^* \times A_1 \times C_{n-1}$	$\mathbb{C}^* \times C_n$	$\mathbb{C}^* \times \mathbb{C}^*$	$1, 3$
$\mathbb{C}^* \times A_n$ $(n \geq 1)$	$\Sigma_a \oplus \Sigma_a^* \oplus (\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$	A_{n-1}	$\mathbb{C}^* \times A_{n-1}$	A_n	$\mathbb{C}^* \times \mathbb{C}^*$	
$\mathbb{C}^* \times A_n \times A_m$ $(n, m \geq 1)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^* \oplus$ $\oplus (\Sigma_a \otimes \omega_1'') \oplus (\Sigma_a \otimes \omega_1'')^*$	$\mathbb{C}^* \times A_{n-1} \times A_{m-1}$	$\mathbb{C}^* \times A_{n-1} \times A_m$	$\mathbb{C}^* \times A_n \times A_{m-1}$	$\mathbb{C}^* \times \mathbb{C}^*$	
$\mathbb{C}^* \times C_n$ $(n \geq 2)$	$\Sigma_a \oplus \Sigma_a^* \oplus (\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$	C_{n-1}	$\mathbb{C}^* \times C_{n-1}$	C_n	$A_1 \times \mathbb{C}^*$	$*$
$\mathbb{C}^* \times C_n \times C_m$ $(n, m \geq 2)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^* \oplus$ $\oplus (\Sigma_a \otimes \omega_1'') \oplus (\Sigma_a \otimes \omega_1'')^*$	$\mathbb{C}^* \times C_{n-1} \times C_{m-1}$	$\mathbb{C}^* \times C_{n-1} \times C_m$	$\mathbb{C}^* \times C_n \times C_{m-1}$	$\mathbb{C}^* \times \mathbb{C}^*$	

Table 5

G	V	H	A	B	N
$\mathbb{C}^* \times \mathrm{SO}_n$ $(n \geq 3)$	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$	$\mathrm{SO}_{n-2} \times \mathbb{Z}_2$	$\mathrm{SO}_{n-2} \times \mathbb{C}^*$	$\mathrm{SO}_{n-1} \rtimes \mathbb{Z}_2$	$\mathbb{C}^* \times (\mathbb{C}^* \rtimes \mathbb{Z}_2)$
D_5	$\omega_4 \oplus \omega_5$	A_3	A_4	B_3	$\mathbb{C}^* \times (\mathbb{C}^* \rtimes \mathbb{Z}_2)$
A_4	$\omega_2 \oplus \omega_2^*$	$A_1 \times A_1$	$A_2 \times A_1$	C_2	$\mathbb{C}^* \times (\mathbb{C}^* \rtimes \mathbb{Z}_2)$
$\mathbb{C}^* \times A_4$	$(\Sigma_a \otimes \omega_2') \oplus (\Sigma_a \otimes \omega_2')^*$	$\mathbb{C}^* \times A_1 \times A_1$	$\mathbb{C}^* \times A_1 \times A_2$	$\mathbb{C}^* \times C_2$	$\mathbb{C}^* \times (\mathbb{C}^* \rtimes \mathbb{Z}_2)$