Lab 7: Matrix Notation in Mathematica

Basic notation

Vectors in mathematica are written as follows:

{element 1, element 2, ... , element n}

You don’t have to distinguish between row and column vectors in Mathematica, since Mathematica uses the context in order to determine which kind of vector you are using.

Matrices in mathematica are written as follows:

{{row 1},{row 2}, ... , {row n}}

For example, a 2x2 matrix is (enter the following):

\[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\]

You can also write this using the “Basic Math Assistant” Palette (drop down menu Palettes - Basic Math Assistant)

\[
\begin{pmatrix}
    a \\
    c
\end{pmatrix}
\begin{pmatrix}
    b \\
    d
\end{pmatrix}
\]

The matrix and vector multiplication that we've been talking about in class is technically known as the "dot product" and is represented by a "." (a period).

\[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\begin{pmatrix}
    e \\
    f
\end{pmatrix}
\]

\[
\begin{pmatrix}
    a \\
    c
\end{pmatrix}
\begin{pmatrix}
    b & d
\end{pmatrix}
\]

[Notice that Mathematica knew to interpret \{e,f\} as a column vector in the first case and as a row vector in the second case.]

Matrix multiplication and addition

Note: If you prefer to see the outputs in a clearer matrix form you can use the MatrixForm[ ] command to convert them.

Write a 2x2 matrix of your choice and multiply it by 3 (a scalar). Scalar multiplication can be accomplished using a space or ".

Add together two 2x2 matrices of your choice. Matrix addition can be accomplished using a +.
Multiply together a 2x2 matrix by a vector of your choice. Matrix multiplication (the dot product) can be accomplished with a ".".

Replace the "." with an "*" or a space to see that only "." corresponds to the standard matrix multiplication.

\[
\text{MatrixForm}[
\begin{pmatrix}
3 & 4 \\
5 & 6
\end{pmatrix}
\times 3
\]
\[
\begin{pmatrix} 9 & 12 \\ 15 & 18 \end{pmatrix}
\]

\[
\text{MatrixForm}[
\begin{pmatrix}
3 & 4 \\
5 & 6
\end{pmatrix}
+ \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}
]
\]
\[
\begin{pmatrix} 7 & 9 \\ 11 & 13 \end{pmatrix}
\]

\[
\text{MatrixForm}[
\begin{pmatrix}
3 & 4 \\
5 & 6
\end{pmatrix}
\cdot \begin{pmatrix} 4 & 5 \end{pmatrix}
]
\]
\[
\begin{pmatrix} 32 \\ 50 \end{pmatrix}
\]

### Matrix Operations in Mathematica

Using the three examples:

\[
m1 = \begin{pmatrix} a, b \\ c, d \end{pmatrix}
\]
\[
\begin{pmatrix} a, b \\ c, d \end{pmatrix}
\]

\[
m2 = \begin{pmatrix} 1, 2 \\ 3, 2 \end{pmatrix}
\]
\[
\begin{pmatrix} 1, 2 \\ 3, 2 \end{pmatrix}
\]

\[
m3 = \begin{pmatrix} 1, -3, 3 \\ 3, -5, 3 \\ 6, -6, 4 \end{pmatrix}
\]
\[
\begin{pmatrix} 1, -3, 3 \\ 3, -5, 3 \\ 6, -6, 4 \end{pmatrix}
\]

Find the transpose using \text{Transpose}[m]

\[
\text{Transpose}[m1]
\]
\[
\begin{pmatrix} a, c \\ b, d \end{pmatrix}
\]

\[
\text{Transpose}[m2]
\]
\[
\begin{pmatrix} 1, 3 \\ 2, 2 \end{pmatrix}
\]

\[
\text{MatrixForm}[\text{Transpose}[m3]]
\]
\[
\begin{pmatrix} 1 & 3 & 6 \\ -3 & -5 & -6 \\ 3 & 3 & 4 \end{pmatrix}
\]

\[
?\text{Transpose}
\]

\[
\text{Transpose}[m3]
\]
\[
\begin{pmatrix} 1, 3, 6 \\ -3, -5, -6 \\ 3, 3, 4 \end{pmatrix}
\]

Find the determinant using \text{Det}[m]
Det gives the determinant of the square matrix \( m \).  

\[ \text{Det}[m3] = 16 \]

Find the inverse of the matrix using \( \text{Inverse}[m] \)

\[ \text{Inverse}[m2] = \{ \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \left\{ \frac{3}{4}, -\frac{1}{4} \right\} \} \]

What does the inverse matrix give you when multiplied by the original matrix (remember to use the dot product)? Does it matter which matrix comes first?

\[ \text{Factor}[m1 . \text{Inverse}[m1]] = \{\{1, 0\}, \{0, 1\}\} \]

(*it should give you the identity matrix*)

---

**Eigenvalues and Eigenvectors**

Find the eigenvalues of the matrix using \( \text{Eigenvalues}[m] \)

\[ \text{Eigenvalues} \]

For example, by entering:

\[ \text{Factor}[\text{Eigenvalues}[m1]] = \frac{1}{2} \left\{ a + d - \sqrt{a^2 + 4 \, b \, c - 2 \, a \, d + d^2} \right\}, \frac{1}{2} \left\{ a + d + \sqrt{a^2 + 4 \, b \, c - 2 \, a \, d + d^2} \right\} \]

\[ \text{Factor}[\text{Eigenvalues}[m2]] = \{4, -1\} \]

\[ \text{Factor}[\text{Eigenvalues}[m3]] = \{4, -2, -2\} \]

Find the eigenvectors of the matrix using \( \text{Eigenvectors}[m] \)

\[ \text{Eigenvectors} \]

For example, by entering:

\[ \text{Factor}[\text{Eigenvectors}[m1]] = \left\{ \left\{ \frac{-a + d + \sqrt{a^2 + 4 \, b \, c - 2 \, a \, d + d^2}}{2 \, c}, 1 \right\}, \left\{ \frac{a - d + \sqrt{a^2 + 4 \, b \, c - 2 \, a \, d + d^2}}{2 \, c}, 1 \right\} \right\} \]
You can find both the eigenvalues and the eigenvectors of a matrix using Eigensystem[m].

?Eigensystem

For example, by entering:

Factor[Eigensystem[m3]]

{{4, -2, -2}, {1, 1, 2}, {-1, 0, 1}, {1, 1, 0}}

For each eigenvector of matrix m3, show that the matrix times the eigenvector equals the eigenvector times its associated eigenvalue. (Don't forget to use the dot product.)

That is, fill in the vector below with the eigenvector and check the answer:

m3 . {1, 1, 2}
{4, 4, 8}
{1, 1, 2} * 4
{4, 4, 8}

Another useful matrix operation is MatrixPower[matrix, t], this takes the matrix to the t power.

For example,

MatrixPower[m2, 3]

{{25, 26}, {39, 38}}

gives the same answer as (enter both and compare):

m2 . m2 . m2

Either way, these commands tell you how much change will occur in a vector over three time steps.

Lab Assignment

Eigenvalues[m3]

{4, -2, -2}

MatrixForm[Eigenvectors[m3]]

\[
\begin{pmatrix}
1 & 1 & 2 \\
-1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

For matrix 3, write down by hand and show the TA:
- the diagonal matrix, D
- the transformation matrix, A

that you would use to write the transition matrix, m3 as A.D.A^{-1}.

DON'T FORGET: The placement of the eigenvalues along the diagonal elements of D must match the placement of the
eigenvectors in *COLUMNS* of matrix A.

Next, write the D and A matrices for matrix 3 in *Mathematica.*

```mathematica
In[4]:= dmat = {{4, 0, 0}, {0, -2, 0}, {0, 0, -2}};
In[5]:= amat = {{1, -1, 1}, {1, 0, 1}, {2, 1, 0}};
```

Recall that Eigenvectors[m3] gives you the eigenvectors of m3 in rows, not in columns as needed for the A matrix. Which matrix operation described above could you use to flip these rows into columns?

```mathematica
In[6]:= Transpose[amat]
Out[6]= {{1, 1, 2}, {-1, 0, 1}, {1, 1, 0}}
```

We showed in class that a matrix, J can be written as $A.D.A^{-1}$. Show that this is true here and that the matrix m3 is equal to amat.dmat.Inverse[amat]

```mathematica
In[7]:= amat.dmat.Inverse[amat]
Out[7]= {{1, -3, 3}, {3, -5, 3}, {6, -6, 4}}
```

```mathematica
In[10]:= m3
Out[10]= {{1, -3, 3}, {3, -5, 3}, {6, -6, 4}}
```

### A caveat

Not all matrices can be diagonalized (i.e., can be written as $A.D.A^{-1}$).

Only nxn matrices that have n independent eigenvectors can be diagonalized. [Independent means that you can't get one of the eigenvectors by simply adding together multiples of the other eigenvectors. Independence implies that the determinant of A will not be zero.]

Here is an example of a matrix that cannot be diagonalized:

```mathematica
In[12]:= m4 = {{-3, 1, -1}, {-7, 5, -1}, {-6, 6, -2}}
Out[12]= {{-3, 1, -1}, {-7, 5, -1}, {-6, 6, -2}}
```

Note that one of the eigenvectors of this matrix is {0,0,0}:

```mathematica
In[13]:= Factor[Eigensystem[m4]]
Out[13]= {{4, -2, -2}, {{0, 1, 1}, {1, 1, 0}, {0, 0, 0}}}
```

m4 is thus known as a "defective" matrix. The only type of square matrix that is defective is one that has a repeated eigenvalue. For example, the eigenvalue of -2 is repeated twice for matrix m4.

Just because an eigenvalue is repeated, however, does not make the matrix defective. For example, matrix 3 also has an eigenvalue of -2 that is repeated twice, but it is possible to find three independent eigenvectors for m3.