Demography Problem: A scientist is studying a population of sticklebacks that generally take two years to reproduce, after which they die. In a pond where the fish are newly introduced, she finds that mortality is high and that only 1/4 survive from year one to year two. A small number of one-year old fish produce offspring (two daughters, on average, per female). If they survive to their second year, however, the number of offspring produced is large (5 daughters per female). She constructs the following Leslie matrix for this population:

$$L = \begin{pmatrix} 2 & 5 \\ 1/4 & 0 \end{pmatrix}$$

(a) Find the eigenvalues and right eigenvectors, $\vec{u}$, of **L**. (b) Using your answers to (a), what is the long-term growth rate and the proportion of the population in each age class at the stable age distribution? (c) Write a matrix, **A**, containing the eigenvectors of **L** as columns. Use your answer to (b) to write the eigenvector associated with the leading eigenvalue as proportions and place this in the first column of **A**. Calculate the inverse matrix, $\mathbf{A}^{-1}$, whose first row gives the left eigenvector associated with the leading eigenvalue. (Check: Make sure that your answer does give a left eigenvector by showing that $\vec{v}\,\mathbf{L} = \lambda\,\vec{v}$.) (d) We can approximate the dynamics of the fish using $\vec{n}(t) \approx \lambda_L^t\, \vec{u}_L \left(\vec{v}_L\, \vec{n}(0)\right)$, where $\vec{u}_L$ is the right eigenvector (written in column format) and $\vec{v}_L$ is the left eigenvector (written in row format) associated with the leading eigenvalue. The scientist knows that the pond started with 10 female fish, all in the first age class [Hint: this gives the initial vector, $\vec{n}(0)$]. Using this information, what is the approximate total population size after 2 years? After 10 years? (e) Compare your answers to (d) to the exact answer obtained by iterating the Leslie matrix, which predicts that the total population size is 152.5 at $t = 2$ and 238,419 at $t = 10$. Explain why the error is so small, even after just two years, but is proportionately worse at $t = 2$. (g) In the long run, would the fish population be larger if it had been started from 10 fish of age one or from 10 fish of age two? Compose an answer using information from the eigenvectors.